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NON-FLAT TOTALLY GEODESIC SURFACES IN SYMMETRIC SPACES OF CLASSICAL TYPE

Katsuya MASHIMO

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Abstract

We give a classification of non-flat totally geodesic surfaces in compact Riemannian symmetric spaces of classical type.

Introduction

Riemannian submanifold M of a Riemannian manifold P is said to be *totally geodesic* if the second fundamental form vanishes everywhere. We confine our consideration to totally geodesic submanifolds in Riemannian symmetric spaces. Totally geodesic submanifolds in Riemannian symmetric spaces have long been studied. Many efforts have been made toward the classification of totally geodesic submanifolds. Important results on construction and classification of totally geodesic submanifolds are summarized in [6]. Recently, a significant progress has been made by Klein. He classified maximal totally geodesic submanifolds in compact Riemannian symmetric spaces of rank 2 in a series of papers [4], [5], [6], [7], [8]. His results are summarized in [6]. In spite of many efforts by many mathematicians, the classification of totally geodesic submanifolds still remains open.

In this paper we focus our attention to totally geodesic submanifolds with smallest dimension. Totally geodesic submanifold of dimension one and flat totally geodesic surfaces are contained in the maximal torus. Thus our targets are totally geodesic surfaces of nonzero constant curvature. As for the totally geodesic submanifold of constant curvature, we mention about the results by Helgason [3] and Nagano and Sumi [11]. Helgason gave the construction of maximal dimensional totally geodesic submanifolds in compact Riemannian symmetric spaces of which curvature is equal to the maximal sectional curvature of the ambient space. The purpose of this paper is to give a classification of non-flat totally geodesic surfaces in compact irreducible Riemannian symmetric spaces.

Let G be a compact simple Lie group and θ be an involutive automorphism of G. We denote by $\mathfrak{g}$ the Lie algebra of G and denote also by θ the differential of θ. Let $\mathfrak{k}$ be the subalgebra of $\mathfrak{g}$ consisting of all elements fixed by θ. Let $\langle , \rangle$ be a bi-invariant inner product of $\mathfrak{g}$ and $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$. Let K be a Lie subgroup of G of which Lie algebra coincides with $\mathfrak{k}$. We identify $\mathfrak{p}$ with the tangent space of the Riemannian symmetric space $P = G/K$. Let M be a totally geodesic submanifold of P emanating from the origin eK of P. The subspace $\mathfrak{m}$ of $\mathfrak{p}$ which is identified with tangent space of M at the origin satisfies

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$$[\mathfrak{m}, [\mathfrak{m}, \mathfrak{m}]] \subset \mathfrak{m}.$$

A subspace $\mathfrak{m}$ of $\mathfrak{p}$ with the above property is called a *Lie triple system*. There exits a one-to-one correspondence between the set of totally geodesic submanifold of P through the origin and the set of Lie triple systems. So the classification of totally geodesic submanifolds of a Riemannian symmetric spaces reduces to the purely algebraic problem. In the former work on the classification of totally geodesic submanifolds of Riemannian symmetric spaces, there are some ways to classify Lie triple system. The method we adopt in this paper is based on the representation theory of $SU(2)$ and we give the explicit expression of the basis of Lie triple systems of totally geodesic surfaces.

Because of the duality, the classification of totally geodesic submanifolds in a compact Riemannian symmetric space P corresponds to those in the noncompact dual of P'. Fujimaru, Kubo and Tamaru [2] studied the classification of totally geodesic surfaces in noncompact Riemannian symmetric space of type AI. Though there is a duality, the tools adopted in [2] and in this paper are completely different. Part of our result is also obtained by Nagatomo, in connection with his study [12].

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1. Notations

We denote by $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Q}$ the field of real, complex and quaternions respectively. We denote by $\overline{z}$ the conjugate of $z \in \mathbb{C}$ and by τ the conjugation of $\mathbb{C}^N$ with respect to $\mathbb{R}^N$; $\tau((z_i)) = (\overline{z_i})$. We denote by $(\ ,\ )$ the standard inner product on $\mathbb{R}^N$ and also by $(\ ,\ )$ the Hermitian inner product $((z_i), (w_i)) = \sum \overline{z_i} w_i$ on $\mathbb{C}^N$. The standard orthonormal basis of $\mathbb{R}^N$ or $\mathbb{C}^N$ will be denoted by $e_1, \cdots, e_N$. We denote by E_{ij} the endomorphism satisfying

$$E_{ij}(e_j) = e_i, \quad E_{ij}(e_k) = 0 \quad (k \neq j),$$

by G_{ij} $(i \neq j)$ the skew-symmetric endomorphism

$$G_{ij}(e_j) = e_i, \quad G_{ij}(e_i) = -e_j, \quad G_{ij}(e_k) = 0 \quad (k \neq i, j),$$

and by S_{ij} the symmetric endomorphism

$$S_{ij}(e_j) = e_i, \quad S_{ij}(e_i) = e_j, \quad S_{ij}(e_k) = 0 \quad (k \neq i, j).$$

We put

$$J = \begin{bmatrix} O & -I_n \\ I_n & O \end{bmatrix} \qquad (N = 2n),$$

$$I_{p,q} = \begin{bmatrix} I_p & O \\ O & -I_q \end{bmatrix} \qquad (N = p + q),$$

$$K_{p,q} = \begin{bmatrix} I_p & O & O & O \\ O & -I_q & O & O \\ O & O & I_p & O \\ O & O & O & -I_q \end{bmatrix} \qquad (N = 2(p+q))$$

where I_n denote the unit matrix of order n.

We denote by 1, i, j, k the standard orthonormal basis of $\mathbb{Q}$. We consider $\mathbb{Q}^n$ as right $\mathbb{Q}$ vector space. A $\mathbb{Q}$-linear endomorphism on $\mathbb{Q}^n$ is expressed by the multiplication of a matrix with coefficients in $\mathbb{Q}$ from the left.

We identify $\mathbb{Q}^n$ with $\mathbb{C}^{2n}$ using the identification $a_0 + a_1\, i + a_2\, j + a_3\, k \in \mathbb{Q}$ with $(a_0 + a_1\, i, a_2 - a_3\, i) \in \mathbb{C}^2$. The right multiplication by j on $\mathbb{Q}^n$ coincides with $\tilde{J} = \tau \circ J = J \circ \tau$ on $\mathbb{C}^{2n}$. A $\mathbb{Q}$-linear endomorphism on $\mathbb{Q}^n$ corresponds to a $\mathbb{C}$-linear endomorphism on $\mathbb{C}^{2n}$ which commutes with $\tilde{J}$.

2. Representations of $SU(2)$

In this section we give a brief review on complex and real irreducible representations of $SU(2)$.

2.1. Complex irreducible representation. The special unitary group $SU(2)$ is the group of matrices which acts on $\mathbb{C}^2$ and leaves invariant the Hermitian inner product $(\ ,\ )$. We can identify $SU(2)$ with the 3-dimensional unit sphere $S^3 \subset \mathbb{C}^2$ by

$$SU(2) \to S^3 : g \mapsto g \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad g \in SU(2).$$

The induced metric on $SU(2)$ is the bi-invariant metric on $SU(2)$. The matrices

$$X_1 = \begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{bmatrix}$$

form a basis of the Lie algebra $\mathfrak{su}(2)$ of $SU(2)$ with

$$[X_1, X_2] = 2\, X_3, \quad [X_2, X_3] = 2\, X_1, \quad [X_3, X_1] = 2\, X_2. \tag{1}$$

If we put

$$H = -\sqrt{-1}\, X_1, \quad X = \frac{1}{2}\left(X_2 - \sqrt{-1}\, X_3\right), \quad Y = \frac{1}{2}\left(-X_2 - \sqrt{-1}\, X_3\right), \tag{2}$$

we have

$$[H, X] = 2\, X, \quad [H, Y] = -2\, Y, \quad [X, Y] = H. \tag{3}$$

Let d be a non-negative integer and let $V(d)$ be the set of all homogeneous polynomial functions on $\mathbb{C}^2$ of degree d. By the contragradient action, $V(d)$ is an irreducible representation of $SU(2)$ of dimension $d + 1$.

Theorem 2.1. *For each positive integer k, there exists a unique complex irreducible representation of $SU(2)$ with* $\dim V = k$.

Let $H,\ X,\ Y$ be a basis of $\mathfrak{su}(2)^{\mathbb{C}}$ satisfying (3) *and let V be a complex irreducible representation of $SU(2)$ with* $\dim V = k$. *With respect to an $SU(2)$-invariant Hermitian inner product on V, there exists an orthonormal basis $\xi_1, \xi_2, \cdots, \xi_k$ of V which satisfy*

$$\left\{\begin{array}{lcl} H \cdot \xi_j & = & (k+1-2j)\,\xi_j \\ X \cdot \xi_j & = & \sqrt{(j-1)(k+1-j)}\,\xi_{j-1} \\ Y \cdot \xi_j & = & \sqrt{j(k-j)}\,\xi_{j+1} \end{array}\right. \tag{4}$$

for $1 \leq j \leq k$ where we put $\xi_0 = \xi_{k+1} = 0$.

Complex irreducible expressions of $SU(2)$ are described in many literatures. But mostly, the basis with respect to which the representation is described is not an orthonormal one. For the proof of Theorem 2.1, we refer to [10].

Let V be a complex irreducible representation of $SU(2)$. If we start from an arbitrary basis which consists of eigenvectors of H, we have the following:

Theorem 2.2. *Let V be an irreducible representation of $SU(2)$ and $(\ ,\ )$ be an $SU(2)$ invariant Hermitian inner product on V. We put $k = \dim V$. Take a basis H, X, Y of $\mathfrak{su}(2)^{\mathbb{C}}$ which satisfy* (3)*. All eigenvalues of the action of H on V are real numbers. We denote them by $\lambda_1, \cdots, \lambda_k$ ($\lambda_1 \geq \cdots \geq \lambda_k$) and let $u_1, \cdots, u_k$ be unit vectors of V satisfying*

$$H \cdot u_i = \lambda_i\, u_i \quad (1 \leq i \leq k).$$

Then
(1) $\lambda_i = k + 1 - 2\,i \quad (1 \leq i \leq k)$,
(2) $u_1, \cdots, u_k$ *forms an orthonormal basis of V,*
(3) *if we put $u_0 = u_{k+1} = 0$ then there exist complex numbers γ_i with*

$$X \cdot u_i = \gamma_{i-1}\, u_{i-1}, \qquad Y \cdot u_i = \overline{\gamma_i}\, u_{i+1}, \tag{5}$$

where

$$|\gamma_i| = \sqrt{i\,(k-i)} \quad (0 \leq i \leq k). \tag{6}$$

Proof. For each element of $\mathfrak{su}(2)^{\mathbb{C}}$, we denote its action on V by the same letter. Since X_1 is a skew-Hermitian endomorphism, all eigenvalues of $H = -\sqrt{-1}\,X_1$ are real numbers. If we put $v_j = Y^{j-1}v_1$ then v_j is an eigenvector of H corresponding to the eigenvalue $\lambda_1 - 2(j-1)$. Since it is easily seen that the subspace $\sum_{j=1}^{\infty} \mathbb{C}v_j$ is an $\mathfrak{su}(2)^{\mathbb{C}}$-invariant subspace, we have $V = \sum_{j=1}^{\infty} \mathbb{C}v_j$. Thus all eigenvalues have multiplicity one and (2) is now obvious. From $H = XY - YX$ we have trace$H = k(\lambda_1 + 1 - k) = 0$, which implies (1).

For each j ($1 \leq j \leq k-1$), we put $Yu_j = c_j u_{j+1}$, $Xu_{j+1} = d_j u_j$. We have

$$c_j \overline{c_j} = \left(Yu_j, Yu_j\right).$$

If we put $C = 1/8\left(X_1{}^2 + X_2{}^2 + X_3{}^2\right)$, using (2) we have

$$\left(Yu_j, Yu_j\right) = \frac{1}{4}\left[\left(\left(X_1{}^2 - 8C\right)u_j, u_j\right) + 2\left(Hu_j, u_j\right)\right].$$

From $8C = -(k^2 - 1)$ (cf. [9, p.295]), we have

$$c_j\overline{c_j} = j(k-j).$$

Similarly we can prove

$$d_j\overline{d_j} = j(k-j).$$

From

$$Hu_1 = (k-1)u_1 = XYu_1 - YXu_1 = c_1d_1u_1$$

we have $c_1d_1 = k-1$ and $d_1 = \overline{c_1}$. Assume that $c_jd_j = j(k-j)$ $(1 \leq j \leq k-2)$ is proved, then by

$$Hu_{j+1} = (k-1-2j)u_{j+1} = XYu_{j+1} - YXu_{j+1} = (c_{j+1}d_{j+1} - c_jd_j)u_{j+1}$$

we have $c_{j+1}d_{j+1} = (j+1)(k-j-1)$ and $d_{j+1} = \overline{c_{j+1}}$. Thus (5) and (6) are proved. □

2.2. Real linear irreducible representation. Let G be a compact connected Lie group. Let (V,ρ) be a complex representation of G and $v_1, \cdots, v_N$ be a basis of V. We denote by $\overline{V}$ the complex conjugate vector space. Namely, $\overline{V}$ itself is an additive group V but the scalar multiplication is defined by $c * x = \overline{c}\,x$ $(c \in \mathbb{C},\ x \in V)$. Define the mapping $S : V \to \overline{V}$ by

$$S\left(\sum_{i=1}^{n} z_i\, v_i\right) = \sum_{i=1}^{n} z_i * v_i = \sum_{i=1}^{n} \overline{z_i}\, v_i$$

and the action $\overline{\rho}$ of $SU(2)$ on $\overline{V}$ so that

$$\overline{\rho} \circ S = S \circ \rho$$

holds. The representation $(\overline{V},\overline{\rho})$ is called the *conjugate* representation of (V,ρ).

A complex irreducible representation (V,ρ) of G is said to be a *self-conjugate* representation if there exists a conjugate-linear automorphism, namely a map $\hat{j} : V \to V$ such that

$$\hat{j}(av + bw) = \bar{a}\hat{j}(v) + \bar{b}\hat{j}(w) \qquad (a, b \in \mathbb{C},\ v, w \in V),$$
$$\hat{j}(\rho(g)v) = \rho(g)\hat{j}(v) \qquad (g \in G,\ v \in V).$$

A conjugate-linear automorphism commuting with ρ is called a *structure map* of (V,ρ).

We denote by $(V_{\mathbb{R}}, \rho_{\mathbb{R}})$ the representation of G over $\mathbb{R}$ obtained by the restriction of the coefficient field from $\mathbb{C}$ to $\mathbb{R}$.

Let (V,ρ) be a self-conjugate representation and $\hat{j}$ be a structure map. By Schur's lemma, $\hat{j}^2 = c$ for some constant $c \neq 0$. It is known that c is a real number and (V,ρ) is said to be of *index* 1 (resp. -1) if $c > 0$ (resp. $c < 0$). If an irreducible representation (V,ρ) of G over $\mathbb{C}$ is a self-conjugate representation of index -1, then $(V_{\mathbb{R}}, \rho_{\mathbb{R}})$ is also an irreducible representation and the complexification of $V_{\mathbb{R}}$ admits a direct sum decomposition

$$V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = (1 + \sqrt{-1}\,\hat{j})V \oplus (1 - \sqrt{-1}\,\hat{j})V,$$

where $(1 + \sqrt{-1}\,\hat{j})V$ is isomorphic to V and $(1 - \sqrt{-1}\,\hat{j})V$ is isomorphic to $\overline{V}$. If (V,ρ) is a self-conjugate representation of index 1, then there exists an irreducible subspace V_0 in $V_{\mathbb{R}}$ with $\dim_{\mathbb{R}} V_0 = \dim_{\mathbb{C}} V$.

For these facts we refer, for instance, to [1], [13].

Now we confine our attention to the case $G = SU(2)$.

Let $\hat{j}$ be a conjugate linear automorphism on $\mathbb{C}^2$ defined by

$$\hat{j}(z, w) = (-\overline{w}, \overline{z}) \quad (z, w \in \mathbb{C}),$$

and extend it to an automorphism on $V(d)$ by

$$(\hat{j}P)(z, w) = \overline{P(\hat{j}(z, w))} \quad (z, w \in \mathbb{C}).$$

The extension $\hat{j}$ is a structure map on $V(d)$ with $\hat{j}^2 = (-1)^d I$.

Lemma 2.3. *If W is an irreducible representation of $SU(2)$ over $\mathbb{R}$, then one of the following holds:*

(i) *The complexification $W^{\mathbb{C}}$ is also an irreducible representation of $SU(2)$ over $\mathbb{C}$. In this case* $\dim W$ *is an odd integer.*

(ii) *The complexification $W^{\mathbb{C}}$ is a reducible representation of $SU(2)$ over $\mathbb{C}$. In this case there exists an irreducible representation V of $SU(2)$ over $\mathbb{C}$ such that $W = V_{\mathbb{R}}$ and* $\dim W$ *is divided by* 4.

Proof. Note that any irreducible representation V of $SU(2)$ is self-conjugate and the index of V is equal to $(-1)^{\dim V - 1}$. The index of V is 1 [resp. -1] if $\dim V$ is odd [resp. even].

If $W^{\mathbb{C}}$ is also irreducible over $\mathbb{C}$, the conjugation of $W^{\mathbb{C}}$ with respect to W is a structure map of index 1. Thus $\dim_{\mathbb{R}} W = \dim_{\mathbb{C}} W^{\mathbb{C}}$ is an odd integer. If $W^{\mathbb{C}}$ is reducible over $\mathbb{C}$, then there exists an irreducible representation V of $SU(2)$ over $\mathbb{C}$, which does not admit any structure map of index 1, such that $W = V_{\mathbb{R}}$ ([13, p.65]). Thus $\dim_{\mathbb{R}} W = 2\dim_{\mathbb{C}} V$ is divided by 4. □

Proposition 2.4. *Let U be a 3-dimensional simple Lie subgroup of $SU(2N)$ and let X_1, X_2 and X_3 be a basis of the Lie algebra $\mathfrak{u}$ of U satisfying* (1). *Put*

$$H = -\sqrt{-1}\, X_1, \quad X = \frac{1}{2}\left(X_2 - \sqrt{-1}\, X_3\right), \quad Y = \frac{1}{2}\left(-X_2 - \sqrt{-1}\, X_3\right).$$

Let $\hat{j}$ be a conjugate linear automorphism on $\mathbb{C}^{2N}$ commuting with the action of U. Assume that

$$X \circ \hat{j} = \varepsilon_2\, \hat{j} \circ Y, \quad \hat{j}^2 = \varepsilon_1 I$$

hold for some constants ε_1 and ε_2 with $\varepsilon_1 = \pm 1$ and $\varepsilon_2 = \pm 1$. If a U-irreducible subspace V of $\mathbb{C}^{2N}$ satisfy $\hat{j}(V) \cap V = \{0\}$, then we have

$$\left(\hat{j}(V), V\right) = 0.$$

Proof. Put $k = \dim V$ and take a basis $\xi_1, \cdots, \xi_k$ of V satisfying (4).

The subspace V is spanned by $Y^{l-1}\xi_1$ $(1 \le l \le k)$ and the subspace $\hat{j}(V)$ is spanned by $\hat{j}(Y^{l-1}\xi_1)$ $(1 \le l \le k)$. Note that

$$(Y u, v) = (u, X v)$$

holds for any u, $v \in \mathbb{C}^{2N}$. We have

$$\left(Y^{l-1}\xi_1, \hat{j}\,(Y^{m-1}\xi_1)\right) = \left(\xi_1, X^{l-1}(\,\hat{j}\,(Y^{m-1}\xi_1))\right) = \varepsilon_1\,(\varepsilon_2)^{l-1}\left(\hat{j}\,\xi_1, Y^{l+m-2}\xi_1\right).$$

Note that $\hat{j}\,\xi_1$ and $Y^{l+m-2}\xi_1$ are eigenvectors of H corresponding to the eigenvalue $1-k$ and $k-1-2(l+m-2) = k+3-2(l+m)$ respectively. Thus, if $l+m \neq k+1$ then we have $\left(\hat{j}\,\xi_1, Y^{l+m-2}\xi_1\right) = 0$.

Assume that $\left(\hat{j}\,\xi_1, Y^{k-1}\xi_1\right) \neq 0$. Since we have

$$\begin{aligned}\left(\hat{j}\,\xi_1, Y^{k-1}\xi_1\right) &= \varepsilon_1\left(\xi_1, \hat{j}\,(Y^{k-1}\xi_1)\right) = \varepsilon_1\,(\varepsilon_2)^{k-1}\left(\xi_1, X^{k-1}(\hat{j}\xi_1)\right)\\ &= \varepsilon_1\,(\varepsilon_2)^{k-1}\left(Y^{k-1}\xi_1, \hat{j}\,\xi_1\right) = \varepsilon_1\,(\varepsilon_2)^{k-1}\overline{\left(\hat{j}\,\xi_1, Y^{k-1}\xi_1\right)},\end{aligned}$$

$\left(\hat{j}\,\xi_1, Y^{n-1}\xi_1\right)$ is either a real number or a pure imaginary number. If α is an arbitrary nonzero complex number, then $\xi_1' = \alpha\,\xi_1, \cdots, \xi_k' = \alpha\,\xi_k$ is also a basis of V satisfying (4). By the argument above with the basis $\xi_1', \cdots, \xi_k'$, we conclude that

$$\left(\hat{j}\,(\alpha\xi_1), Y^{k-1}(\alpha\xi_1)\right) = \alpha^2\left(\hat{j}\,\xi_1, Y^{k-1}\xi_1\right)$$

is also a real number or a pure imaginary number, which is a contradiction. Thus we have

$$\left(\hat{j}\,\xi_1, Y^{k-1}\xi_1\right) = 0.$$

□

3. Totally geodesic surfaces

Let G be a compact, connected and simple Lie group of classical type. Let θ be an involutive automorphism on G and let K be the set of elements of G invariant under θ. We denote by $\mathfrak{g}$, $\mathfrak{k}$ the Lie algebras of G and K respectively and denote also by θ the differential of θ. We denote by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the bi-invariant metric.

Let M be a non-flat totally geodesic surface of $P = G/K$ and $\mathfrak{m} = T_{eK}M \subset \mathfrak{p}$ be the corresponding Lie triple system. Because $\mathfrak{m}$ is a Lie triple system and is non-abelian, if we put $\mathfrak{u} = [\mathfrak{m}, \mathfrak{m}] + \mathfrak{m}$ then $\mathfrak{u}$ is a 3-dimensional Lie subalgebra of $\mathfrak{g}$, which is isomorphic to $\mathfrak{su}(2)$. Let $(\,,)$ be a G-invariant inner product on $\mathfrak{g}$. Let X_2', X_3' be an orthonormal basis of $\mathfrak{m}(\subset \mathfrak{g})$ with respect to $(\,,)$. If we put $X_1' = [X_2', X_3']$, we have $\left([X_1', X_2'], X_1'\right) = \left([X_1', X_2'], X_2'\right) = 0$, thus $[X_1', X_2']$ is proportional to X_3'. Similarly, $[X_1', X_3']$ is proportional to X_2'. From $\left([X_1', X_2'], X_3'\right) + \left([X_1', X_3'], X_2'\right) = 0$, there exists a constant c such that

$$[[X_2', X_3'], X_2'] = cX_3',\ [[X_2', X_3'], X_3'] = -cX_2'.$$

From $c\left(X_3', X_3'\right) = \left([[X_2', X_3'], X_2'], X_3'\right) = \left([X_2', X_3'], [X_2', X_3']\right) > 0$, c is a positive constant. If we put $X_2 = \dfrac{2}{\sqrt{c}}X_2$, $X_3 = \dfrac{2}{\sqrt{c}}X_3$, then we have

$$[[X_2, X_3], X_2] = 4X_3, \quad [[X_2, X_3], X_3] = -4X_3. \tag{7}$$

In our classification, we shall describe a totally geodesic surface by giving the basis of the corresponding Lie triple system satisfying (7).

3.1. Type *AI* **:** $SU(n)/SO(n)$**.** Let θ be the involutive automorphism on $G = SU(n)$ defined by

$$\theta(g) = \tau \circ g \circ \tau \qquad (g \in SU(n)),$$

and put

$$K = \{g \in SU(n) : \theta(g) = g\} = SO(n).$$

Note that $\theta([g_{ij}]) = [\bar{g}_{ij}]$.

Theorem 3.1. *Let M be a non-flat totally geodesic surface of $SU(n)/SO(n)$, $\mathfrak{m}$ be the corresponding Lie triple system and U be the connected Lie subgroup of $SU(n)$ with Lie algebra $\mathfrak{u} = [\mathfrak{m}, \mathfrak{m}] + \mathfrak{m}$. Take a basis X_2, X_3 of $\mathfrak{m}$ satisfying* (7).

(i) *There exist an orthogonal direct sum decomposition*

$$\mathbb{C}^n = V_0 \oplus V_1 \oplus \cdots \oplus V_k,$$

where V_0 is a trivial $\mathfrak{u}$-module and V_i $(i = 1, \cdots, k)$ are τ-invariant and $\mathfrak{u}$-irreducible subspaces with $\dim V_i \geq 2$.
(ii) *Corresponding to the direct sum decomposition in* (i)*, the matrix representations of X_2 and X_3 are decomposed into blocks. Thus, without loss of generality, we assume $\mathbb{C}^n = V_1$. There exists an orthonormal basis $u_1, \cdots, u_n$ of $\mathbb{C}^n$ satisfying $\tau(u_i) = u_i$ and $g = [u_1, \cdots, u_n] \in SO(n)$ such that*

$$\mathrm{Ad}(g)X_2 = \sqrt{-1} \sum_{i=1}^{n} (n + 1 - 2\,i)\, E_{i,i} \tag{8}$$

$$\mathrm{Ad}(g)X_3 = -\sqrt{-1} \left[\sum_{i=1}^{n-2} \sqrt{i(n-i)} S_{i,i+1} + \varepsilon \sqrt{n-1} S_{n-1,n} \right] \tag{9}$$

where

$$\varepsilon = \begin{cases} 1 & \text{if } n \equiv 1 \ (\mathrm{mod} 2), \\ \pm 1 & \text{if } n \equiv 0 \ (\mathrm{mod} 2). \end{cases}$$

Proof. We put $X_1 = \dfrac{1}{2}[X_2, X_3]$ and

$$H = -\sqrt{-1}\, X_2, \quad X = \frac{1}{2}\left(X_1 + \sqrt{-1}\, X_3\right), \quad Y = \frac{1}{2}\left(-X_1 + \sqrt{-1}\, X_3\right).$$

(i) We shall show that if we take a τ-invariant and U-invariant subspace V of $\mathbb{C}^n$, then for any U-irreducible subspace V' in V there exists a τ-invariant, U-irreducible subspace $V_1 \subset V$ which is isomorphic to V'. Then we obtain (i) by induction.

Let V be a τ-invariant and U-invariant subspace of $\mathbb{C}^n$, and V' be a U-irreducible subspace in V. We put $k = \dim V'$ and take a basis $\xi_1, \cdots, \xi_k$ of V' satisfying (4). Since $\tau(V')$ is also a U-irreducible subspace, $\tau(V') \cap V'$ is either V' or $\{0\}$. If $\tau(V') \cap V' = V'$ holds then V' is τ-invariant, and we are done. So we now suppose $\tau(V') \cap V' = \{0\}$. If we put $y_i = \sqrt{-1}\,(\xi_i - \tau(\xi_i))$, then y_i's are nonzero vectors. Since

$$\tau \circ H = H \circ \tau, \quad \tau \circ X = X \circ \tau, \quad \tau \circ Y = Y \circ \tau$$

hold, the subspace

$$V_1 = \oplus_{i=1}^{k} \mathbb{C}\, y_i$$

is a τ-invariant and U-irreducible subspace of V.

(ii) Assume that the action of U on $\mathbb{C}^n$ is irreducible.

We denote also by H the representation matrix of H with respect to the standard orthonormal base. Since H is a real symmetric matrix, there exists a matrix $g_1 \in SO(n)$ such that

$$\mathrm{Ad}(g_1)H = \mathrm{diag}(\lambda_1, \cdots, \lambda_n) \qquad (\lambda_1 \geq \cdots \geq \lambda_n).$$

Then by Theorem 2.2, we have $\lambda_i = n + 1 - 2i$ $(1 \leq i \leq n)$. By definition, X and Y are real matrices, and again by Theorem 2.2, we have

$$\mathrm{Ad}(g_1)X = \sum_{i=1}^{n-1} \gamma_i\, E_{i,i+1}, \quad \mathrm{Ad}(g_1)Y = \sum_{i=1}^{n-1} \gamma_i\, E_{i+1,i},$$

where $\gamma_i = \pm\sqrt{i\,(n-i)}$ $(1 \leq i \leq n)$.

If $g_2 \in SO(n)$ satisfy $\mathrm{Ad}(g_2)\mathrm{Ad}(g_1)H = \mathrm{Ad}(g_1)H$ then

$$g_2 = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) \qquad \left(\varepsilon_i = \pm 1,\ \prod_{i=1}^{n} \varepsilon_i = 1\right).$$

If we put $g = g_2\, g_1$ then

$$\mathrm{Ad}(g)X = \sum_{i=1}^{n-1} \varepsilon_i\, \varepsilon_{i+1}\, \gamma_i\, E_{i,i+1}, \quad \mathrm{Ad}(g)Y = \sum_{i=1}^{n-1} \varepsilon_i\, \varepsilon_{i+1}\, \gamma_i\, E_{i+1,i}.$$

If n is an odd integer, there exist $\varepsilon_1, \cdots, \varepsilon_n$ such that

$$\varepsilon_1\, \varepsilon_2\, \gamma_1 = \sqrt{n-1},\ \cdots,\ \varepsilon_{n-2}\, \varepsilon_{n-1}\, \gamma_{n-2} = \sqrt{2(n-2)}, \quad \varepsilon_{n-1}\, \varepsilon_n\, \gamma_{n-1} = \sqrt{n-1},$$

and, if n is an even integer, there exist $\varepsilon_1, \cdots, \varepsilon_n$ such that

$$\varepsilon_1\, \varepsilon_2\, \gamma_1 = \sqrt{n-1},\ \cdots,\ \varepsilon_{n-2}\, \varepsilon_{n-1}\, \gamma_{n-2} = \sqrt{2(n-2)}, \quad \varepsilon_{n-1}\, \varepsilon_n\, \gamma_{n-1} = \pm\sqrt{n-1}.$$

From $X_2 = \sqrt{-1}\, H$ and $X_3 = -\sqrt{-1}\,(X + Y)$, we obtain (8) and (9). □

EXAMPLE 1. Let M be a non-flat totally geodesic surface of $SU(4)/SO(4)$ and $\mathfrak{m}$ be the corresponding Lie triple system. If the action of the Lie algebra $\mathfrak{u} = [\mathfrak{m}, \mathfrak{m}] + \mathfrak{m}$ on $\mathbb{C}^4$ is irreducible, then, by Theorem 3.1, there exists an element $k \in SO(4)$ such that $\mathrm{Ad}(k)\mathfrak{m}$ coincides with one of the following;

(i) $\mathfrak{m} = \mathbb{R}X_2 + \mathbb{R}X_{3,1}$,

(ii) $\mathfrak{m}' = \mathbb{R}X_2 + \mathbb{R}X_{3,-1}$,

where

$$X_2 = \sqrt{-1}\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}, \quad X_{3,\varepsilon} = -\sqrt{-1}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \varepsilon\sqrt{3} \\ 0 & 0 & \varepsilon\sqrt{3} & 0 \end{bmatrix} \ (\varepsilon = \pm 1).$$

Proposition 3.2. *Lie triple systems* $\mathfrak{m}$ *and* $\mathfrak{m}'$ *given in* Example *1 are not congruent under the action of* $SO(4)$.

Proof. Assume that there exists $k \in SO(4)$ such that $\mathrm{Ad}(k)\mathfrak{m} = \mathfrak{m}'$. If we put $X_1 = \frac{1}{2}[X_2, X_{3,-1}] \in \mathfrak{so}(4)$, then $\exp_{ad(tX_1)}$ acts as rotation on $\mathfrak{m}'$ through an angle t and as identity on the orthogonal complement of $\mathfrak{m}'$. Thus there exists $k' \in SO(4)$ with $\mathrm{Ad}(k')\mathfrak{m}' = \mathfrak{m}'$ and $\mathrm{Ad}(k')(\mathrm{Ad}(k)X_2) = X_2$. If we put $u = k'k \in SO(4)$, we have $\mathrm{Ad}(u)\mathfrak{m} = \mathfrak{m}'$ and $\mathrm{Ad}(u)X_2 = X_2$. Since u commutes with X_2, u is a diagonal matrix. Since $\mathrm{Ad}(u)X_3$ is orthogonal to $X_2 = \mathrm{Ad}(u)X_2$, either $\mathrm{Ad}(u)X_3 = X_{3,-1}$ or $\mathrm{Ad}(u)X_3 = -X_{3,-1}$ holds. In either case, we have $\det(u) = -1$, which is a contradiction. □

Example 2. Let M be a non-flat totally geodesic surface of $SU(7)/SO(7)$ and $\mathfrak{m}$ be the corresponding Lie triple system. Assume that the decomposition of $\mathbb{C}^7$ given in Theorem 3.1 (i) is

$$\mathbb{C}^7 = \mathbb{C}^3 \oplus \mathbb{C}^4.$$

For $X \in \mathfrak{su}(3)$ and $Y \in \mathfrak{su}(4)$ put $M(X,Y) = \left[\begin{array}{c|c} X & \\ \hline & Y \end{array}\right]$. There exists an element $g \in SU(7)$ such that $\mathrm{Ad}(g)\mathfrak{m} \subset \{M(X,Y) \mid X \in \mathfrak{su}(3), Y \in \mathfrak{su}(4)\}$. By Theorem 3.1 (ii), there exists an element $k \in SO(3) \times SO(4)$ such that

$$\mathrm{Ad}(k)\mathfrak{m} = \mathbb{R}M(Y_2, X_2) + \mathbb{R}M(Y_3, X_{3,\varepsilon}),$$

where X_2, $X_{3,\epsilon}$ are those defined in Example 1 and

$$Y_2 = \sqrt{-1}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad Y_3 = \sqrt{-1}\begin{bmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{bmatrix}.$$

Lie triple systems $\mathbb{R}M(Y_2, X_2) + \mathbb{R}M(Y_3, X_{3,1})$ and $\mathbb{R}M(Y_2, X_2) + \mathbb{R}M(Y_3, X_{3,-1})$ are congruent under the action $\mathrm{Ad}(\mathrm{diag}(1, 1, 1, -1, -1, -1, -1))$.

3.2. Type AII : $SU(2n)/Sp(n)$**.** Let τ be the conjugation of $\mathbb{C}^{2n}$ with respect to $\mathbb{R}^{2n}$ and put $\tilde{J} = J \circ \tau$. Let θ be the involutive automorphism on $SU(2n)$ defined by

$$\theta(g) = \tilde{J} \circ g \circ \tilde{J}^{-1} = J \circ \overline{g} \circ J^{-1}$$

and put

$$K = \{g \in SU(2n) : \theta(g) = g\} = Sp(n).$$

Theorem 3.3. *Let M be a non-flat totally geodesic surface of $SU(2n)/Sp(n)$,* $\mathfrak{m}$ *be the corresponding Lie triple system and U be the connected Lie subgroup of $SU(2n)$ with Lie algebra* $\mathfrak{u} = [\mathfrak{m}, \mathfrak{m}] + \mathfrak{m}$. *Take a basis X_2, X_3 of* $\mathfrak{m}$ *satisfying* (7).

(i) *There exist an orthogonal direct sum decomposition*

$$\mathbb{C}^{2n} = \mathbb{Q}^n = V_0 \oplus V_1 \oplus \tilde{J}V_1 \oplus \cdots \oplus V_k \oplus \tilde{J}V_k.$$

where V_0 is a trivial $\mathfrak{u}$-module and V_i ($i = 1, \cdots, k$) are U-irreducible subspaces of $\mathfrak{u}$ with $\dim V_i \geq 2$.

(ii) *Corresponding to the direct sum decomposition in* (i)*, the matrix representations of X_2 and X_3 are decomposed into blocks. Thus, without loss of generality, we assume that $\mathbb{C}^{2n} = V_1 \oplus \widetilde{J}V_1$ and $\left(V_1, \widetilde{J}V_1\right) = \{0\}$. There exists an element $g = [u_1, \cdots, u_n, \tilde{J}u_1, \cdots, \tilde{J}u_n] \in Sp(n)$, such that*

$$\mathrm{Ad}(g)(X_2) = \sum_{i=1}^{n-1} \sqrt{i(n-i)}\,(G_{i,i+1} - G_{n+i,n+i+1})\,, \tag{10}$$

$$\mathrm{Ad}(g)(X_3) = \sqrt{-1} \sum_{i=1}^{n-1} \sqrt{i(n-i)}\,(S_{i,i+1} + S_{n+i,n+i+1})\,. \tag{11}$$

Proof. Let $X_1 = 1/2\,[X_2, X_3]$ and put

$$H = -\sqrt{-1}X_1,\ X = \frac{1}{2}(X_2 - \sqrt{-1}X_3),\ Y = \frac{1}{2}(-X_2 - \sqrt{-1}X_3).$$

We have

$$\tilde{J} \circ H = -H \circ \tilde{J}, \quad \tilde{J} \circ X = Y \circ \tilde{J}, \quad \tilde{J} \circ Y = X \circ \tilde{J}.$$

(i) We shall show that if we take a $\tilde{J}$-invariant and U-invariant subspace V of $\mathbb{C}^n$, then for any U-irreducible subspace V_1 in V we have $\left(\tilde{J}V_1, V_1\right) = 0$. Note that $V \cap \left(\tilde{J}V_1 \oplus V_1\right)^{\perp}$ is a $\tilde{J}$-invariant and U-invariant subspace. Thus we obtain (i) by induction.

Let V be a $\tilde{J}$-invariant and U-invariant subspace of $\mathbb{C}^n$ and $V_1 \subset V$ be a U-irreducible subspace. Since $\tilde{J}V_1$ is a U-invariant subspace in V, $\tilde{J}V_1 \cap V_1$ is either V_1 or $\{0\}$. Assume that $\tilde{J}V_1 = V_1$ holds. The dimension of V_1 is an even integer, for $\tilde{J}$ is a structure map on V_1 of index -1. Put $k = \dim V = 2k'$ and take an orthonormal basis $\xi_1, \cdots, \xi_k$ of V_1 satisfying (4). From $H \circ \tilde{J} = -\tilde{J} \circ H$ we conclude that there exist unit complex numbers α_i ($1 \leq i \leq k'$) with

$$\tilde{J}\xi_i = \alpha_i\,\xi_{k+1-i}, \quad \tilde{J}\xi_{k+1-i} = -\alpha_i\,\xi_i.$$

From $\tilde{J} \circ Y = X \circ \tilde{J}$, we have

$$\left(\xi_{k'}, \tilde{J}(Y\,\xi_{k'})\right) = -k'\,\alpha_{k'} = \left(\xi_{k'}, X(\tilde{J}\,\xi_{k'})\right) = k'\,\alpha_{k'},$$

which is a contradiction. Thus we have $\tilde{J}V_1 \cap V_1 = \{0\}$. By Proposition 2.4, we have

$$\left(\tilde{J}V_1, V_1\right) = 0.$$

(ii) Let V be a U-irreducible subspace of $\mathbb{C}^{2n}$ with

$$\mathbb{C}^{2n} = V \oplus \tilde{J}V, \quad \left(\tilde{J}V, V\right) = 0.$$

Take an orthonormal basis $\xi_1, \cdots, \xi_n$ of V satisfying (4) (replacing k by n). Since $\xi_1, \cdots, \xi_n$,

$\tilde{J}\xi_1, \cdots, \tilde{J}\xi_n$ is an orthonormal basis of $\mathbb{C}^{2n}$, there exists an element $g \in Sp(n)$ such that

$$g^{-1}(e_i) = \xi_i, \quad g^{-1}(e_{n+i}) = \tilde{J}\xi_i \quad (1 \le i \le n).$$

From $X \circ \tilde{J} = \tilde{J} \circ Y$ and $Y \circ \tilde{J} = \tilde{J} \circ X$ we have

$$\begin{array}{rcll} X(\tilde{J}\xi_i) & = & \sqrt{i(n-i)}\,\tilde{J}\xi_{i+1} & (1 \le i \le n-1), \\ X(\tilde{J}\xi_n) & = & 0, & \\ Y(\tilde{J}\xi_{i+1}) & = & \sqrt{i(n-i)}\,\tilde{J}\xi_i & (1 \le i \le n-1), \\ Y(\tilde{J}\xi_1) & = & 0. & \end{array}$$

Namely we have

$$\begin{aligned} \mathrm{Ad}(g)X &= \sum_{i=1}^{n-1} \sqrt{i(n-i)}\,E_{i,i+1} + \sum_{i=1}^{n-1} \sqrt{i(n-i)}\,E_{n+i+1,n+i}, \\ \mathrm{Ad}(g)Y &= \sum_{i=1}^{n-1} \sqrt{i(n-i)}\,E_{i+1,i} + \sum_{i=1}^{n-1} \sqrt{i(n-i)}\,E_{n+i,n+i+1}. \end{aligned}$$

From $X_2 = X - Y$ and $X_3 = \sqrt{-1}\,(X+Y)$, we obtain (10) and (11). □

Example 3. The subspace spanned by the following matrices is a Lie triple system of $SU(6)/Sp(3)$;

$$X_2 = \left[\begin{array}{ccc|ccc} 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ -\sqrt{2} & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & -\sqrt{2} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -\sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & -\sqrt{2} \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \end{array}\right],$$

$$X_3 = \sqrt{-1}\left[\begin{array}{ccc|ccc} 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \end{array}\right].$$

3.3. Type AIII : $SU(p+q)/S(U(p) \times U(q))$. Let θ be the involutive automorphism on $G = SU(p+q)$ defined by

$$\theta(g) = I_{p,q} \circ g \circ I_{p,q}$$

and put

$$K = \{g \in SU(p+q) : \theta(g) = g\} = S(U(p) \times U(q)).$$

Theorem 3.4. *Let M be a non-flat totally geodesic surface of $SU(p+q)/S(U(p)\times U(q))$, $\mathfrak{m}$ be the corresponding Lie triple system and U be the connected Lie subgroup of $SU(p+q)$ with Lie algebra $\mathfrak{u} = [\mathfrak{m},\mathfrak{m}] + \mathfrak{m}$. Take a basis X_2, X_3 of $\mathfrak{m}$ satisfying* (7).
(i) *There exist an orthogonal direct sum decomposition*

$$\mathbb{C}^{p+q} = V_0 \oplus V_1 \oplus \cdots \oplus V_k,$$

where V_0 is a trivial $\mathfrak{u}$-module and V_i $(i = 1,\cdots,k)$ are $I_{p,q}$-invariant and $\mathfrak{u}$-irreducible subspaces with $\dim V_i \geq 2$.
(ii) *If V is an $I_{p,q}$-invariant, U-irreducible subspace of $\mathbb{C}^{p+q}$, then we have*

$$\left|\dim\{v \in V : I_{p,q}(v) = v\} - \dim\{v \in V : I_{p,q}(v) = -v\}\right| \leq 1.$$

(iii) *Corresponding to the direct sum decomposition in* (i)*, the matrix representations of X_2 and X_3 are decomposed into blocks. Thus, without loss of generality, we assume that $\mathbb{C}^{p+q} = V_1$. There exists an element $g = [u_1,\cdots,u_{p+q}] \in S(U(p)\times U(q))$ such that*

$$\mathrm{Ad}(g)X_2 = \sum_{i=1}^{q}\sqrt{(2i-1)(p+q+1-2i)}\,G_{i,p+i} + \sum_{i=1}^{p-1}\sqrt{2i(p+q-2i)}\,G_{p+i,i+1}, \tag{12}$$

$$\mathrm{Ad}(g)X_3 = \sqrt{-1}\left[\sum_{i=1}^{q}\sqrt{(2i-1)(p+q+1-2i)}\,S_{p+i,i} + \sum_{i=1}^{p-1}\sqrt{2i(p+q-2i)}\,S_{i+1,p+i}\right]. \tag{13}$$

Proof. Put $X_1 = 1/2\,[X_2, X_3]$ and

$$H = -\sqrt{-1}X_1,\ X = \frac{1}{2}(X_2 - \sqrt{-1}X_3),\ Y = -\frac{1}{2}(X_2 + \sqrt{-1}X_3) = {}^t\overline{X}.$$

From

$$I_{p,q}\circ X_1 = X_1\circ I_{p,q},\quad I_{p,q}\circ X_i = -X_i\circ I_{p,q}\quad (i = 2,3),$$
$$[X_1, X_2] = 2X_3,\quad [X_2, X_3] = 2X_1,\quad [X_3, X_1] = 2X_2,$$

we have

$$I_{p,q}\circ H = H\circ I_{p,q},\quad I_{p,q}\circ X = -X\circ I_{p,q},\quad I_{p,q}\circ Y = -Y\circ I_{p,q}. \tag{14}$$

(i) We shall show that if we take an $I_{p,q}$-invariant and U-invariant subspace V, then there exists an $I_{p,q}$-invariant and U-irreducible subspace $V_1 \subset V$. Note that $V \cap \left(I_{p,q}V_1\right)^{\perp}$ is an $I_{p,q}$-invariant and U-invariant subspace. Thus we obtain (i) by induction.

Let V be an $I_{p,q}$-invariant and U-invariant subspace of $\mathbb{C}^{p+q}$ and take an U-irreducible subspace $V' \subset V$. Since $I_{p,q}V'$ is a U-invariant subspace, $I_{p,q}(V')\cap V'$ is either $\{0\}$ or V'. Assume that $I_{p,q}(V')\cap V' = \{0\}$ holds. Put $\dim V' = n$ and take an orthonormal basis $\xi_1,\cdots$, ξ_n of V' satisfying (4) (replacing k by n). We put

$$v_i = \frac{1}{\sqrt{2}}\left(1 + (-1)^i I_{p,q}\right)\xi_i \quad (1 \leq i \leq n).$$

Note that $v_i \neq 0$ from $I_{p,q}(V')\cap V' = \{0\}$. From (14), we have

$$\begin{aligned}
H v_i &= \frac{1}{\sqrt{2}}\left(1+(-1)^i I_{p,q}\right) H\xi_i = (n+1-i)v_i,\\
X v_i &= \frac{1}{\sqrt{2}}\left(1+(-1)^{i-1} I_{p,q}\right) X\xi_i = \sqrt{(i-1)(n+1-i)}v_{i-1},\\
Y v_i &= \frac{1}{\sqrt{2}}\left(1+(-1)^{i+1} I_{p,q}\right) Y\xi_i = \sqrt{i(n-i)}v_{i+1}.
\end{aligned}$$

Thus

$$V_1 = \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_n$$

is an $I_{p,q}$-invariant and U-irreducible subspaces of V.

(ii) Assume that the action of U on $\mathbb{C}^{p+q}$ is irreducible.

There exists an element $g \in K = S(U(p) \times U(q))$ such that $\mathrm{Ad}(g)H$ is contained in the maximal torus of $\mathfrak{k}$. Namely, we can take an element $g \in S(U(p) \times U(q))$ such that

$$\mathrm{Ad}(g)H = \mathrm{diag}(a_1, \cdots, a_p; b_1, \cdots, b_q)$$

where $a_1 > \cdots > a_p$ and $b_1 > \cdots > b_q$ hold.

We denote by ξ_i the i-th column vector of g^{-1}. Then, by Theorem 2.2, the set $\{a_1, \cdots, a_p, b_1, \cdots, b_q\}$ coincides with $\{p+q-1,\ p+q-2,\ \cdots,\ 1-p-q\}$.

Without loss of generality, we may assume $a_1 > b_1$.

- Since a_1 is the largest eigenvalue of H $a_1 = p+q-1$. We have $I_{p,q}\xi_1 = \xi_1$ and $H \cdot \xi_1 = (p+q-1)\xi_1$.
- From $I_{p,q} \circ Y = -Y \circ I_{p,q}$, we have $I_{p,q}(Y \cdot \xi_1) = -Y \cdot \xi_1$ and from $[H, Y] = -2Y$ we have $H(Y \cdot \xi_1) = (p+q-3)\, Y \cdot \xi_1$. Thus we have $b_1 = p+q-3$ and there exists a complex number γ_i with

$$Y \cdot \xi_1 = \gamma_1\, \xi_{p+1}, \quad |\gamma_1| = \sqrt{p+q-1}.$$

- Similarly we have

$$Y \cdot \xi_{p+1} = \gamma_2\, \xi_2, \quad |\gamma_2| = \sqrt{2(p+q-2)}$$

  etc.

Finally we have $p-q = 0, 1$ and the matrix representation of Y with respect to the basis $\xi_1, \cdots, \xi_p, \xi_{p+1}, \cdots, \xi_{p+q}$ is

$$\mathrm{Ad}(g)Y = \sum_{i=1}^{q} \gamma_{2i-1}\, E_{p+i,i} + \sum_{i=1}^{p-1} \gamma_{2i}\, E_{i+1,p+i}.$$

(iii) An element g' of $SU(p+q)$ satisfying $\mathrm{Ad}(g')\mathrm{Ad}(g)H = \mathrm{Ad}(g)H$ is of the following form

$$g' = \mathrm{diag}(\varepsilon_1, \cdots, \varepsilon_p, \varepsilon_{p+1}, \cdots, \varepsilon_{p+q}), \quad |\varepsilon_i| = \pm 1,\ \prod_{i=1}^{p+q} \varepsilon_i = 1.$$

We can choose $\varepsilon_1, \cdots, \varepsilon_{p+q}$ so that all of the coefficients of $\mathrm{Ad}(g')\mathrm{Ad}(g)(Y)$ are non-negative numbers. Thus there exists an element $g \in S(U(p) \times U(q))$ such that

$$\mathrm{Ad}(g)(Y) = \sum_{i=1}^{q} \sqrt{(2i-1)(p+q+1-2i)}\, E_{p+i,i} + \sum_{i=1}^{p-1} \sqrt{2i(p+q-2i)}\, E_{i+1,p+i}.$$

From

$$X_2 = {}^t\overline{Y} - Y, \quad X_3 = \sqrt{-1}\left({}^t\overline{Y} + Y\right),$$

we obtain (12) and (13). □

EXAMPLE 4. The subspace spanned by the following matrices is a Lie triple system of $SU(5)/S(U(3) \times U(2))$

$$X_2 = \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -\sqrt{6} & \sqrt{6} \\ 0 & 0 & 0 & 0 & -2 \\ \hline -2 & \sqrt{6} & 0 & 0 & 0 \\ 0 & -\sqrt{6} & 2 & 0 & 0 \end{array}\right], \quad X_3 = \sqrt{-1}\left[\begin{array}{ccc|cc} 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{6} & \sqrt{6} \\ 0 & 0 & 0 & 0 & 2 \\ \hline 2 & \sqrt{6} & 0 & 0 & 0 \\ 0 & \sqrt{6} & 2 & 0 & 0 \end{array}\right].$$

3.4. Type CI : $Sp(n)/U(n)$. Let θ be the involutive automorphism on $Sp(n)$ defined by

$$\theta(g) = J \circ g \circ J^{-1} = \overline{g}$$

and put

$$K = \{g \in Sp(n) : \theta(g) = g\} = U(n).$$

Theorem 3.5. *Let M be a non-flat totally geodesic surface of $Sp(n)/U(n)$, $\mathfrak{m}$ be the corresponding Lie triple system and U be the connected Lie subgroup of $Sp(n)$ with Lie algebra $\mathfrak{u} = [\mathfrak{m}, \mathfrak{m}] + \mathfrak{m}$. Take a basis X_2, X_3 of $\mathfrak{m}$ satisfying* (7).

(i) *There exist an orthogonal direct sum decomposition*

$$\mathbb{C}^{2n} = \mathbb{Q}^n = V_0 \oplus V_1 \oplus \cdots \oplus V_k,$$

where V_0 is a trivial $\mathfrak{u}$-module and V_i $(i = 1, \cdots, k)$ are J-invariant and U-irreducible subspaces of $\mathfrak{u}$ with $\dim V_i \geq 2$.

(ii) *Corresponding to the direct sum decomposition in* (i)*, the matrix representations of X_2 and X_3 are decomposed into blocks. Thus, without loss of generality, we assume that $\mathbb{C}^{2n} = V_1$. Put*

$$(15) \qquad \widetilde{X}_2 = \sqrt{-1} \sum_{i=1}^{n} (2n+1-2i)\left(E_{i,i} - E_{n+i,n+i}\right),$$

$$(16) \qquad \widetilde{X}_{3,\varepsilon} = \sqrt{-1}\left[\sum_{i=1}^{n-1} \sqrt{i(2n-i)}\left(S_{i,i+1} - S_{n+i,n+i+1}\right) + \varepsilon n\, S_{n,2n}\right] \quad (\varepsilon = \pm 1).$$

There exists an element $g = [u_1, \cdots, u_n, J u_1, \cdots, J u_n] \in U(n)$ such that

$$\mathrm{Ad}(g)X_2 = \widetilde{X}_2, \quad \mathrm{Ad}(g)X_3 = \widetilde{X}_{3,\varepsilon}.$$

Lie triple system $\mathbb{R}\widetilde{X}_2 + \mathbb{R}\widetilde{X}_{3,1}$ and $\mathbb{R}\widetilde{X}_2 + \mathbb{R}\widetilde{X}_{3,-1}$ are not congruent under the action of $U(n)$.

Proof. (i) We shall show that if we take a J-invariant and U-invariant subspace V of $\mathbb{C}^{2n}$, then there exists a J-invariant and U-irreducible subspace $V_1 \subset V$. Note that $V \cap (JV_1)^{\perp}$ is a J-invariant and U-invariant subspace. Thus we obtain (i) by induction.

In the proof of (i), We use the following basis

$$H = -\sqrt{-1}X_1,\ X = \frac{1}{2}(X_2 + \sqrt{-1}\,X_3),\ Y = \frac{1}{2}(-X_2 + \sqrt{-1}\,X_3)$$

of $\mathfrak{u}^{\mathbb{C}}$ which satisfy

$$\begin{gathered}[H, X] = 2\,X, \quad [H, Y] = -2\,Y, \quad [X, Y] = H, \\ J \circ H = H \circ J, \quad J \circ X = -X \circ J, \quad J \circ Y = -Y \circ J.\end{gathered}$$

Let V be a J-invariant and U-invariant subspace of $\mathbb{C}^{2n}$ and V' be a U-irreducible subspace of V. Since $J(V')$ is a U-invariant subspace, $J(V') \cap V'$ coincides with V' or $\{0\}$. Assume that $J(V') \cap V' = \{0\}$. We put $k = \dim V'$ and take an orthonormal basis $\xi_1, \cdots, \xi_k$ of V' satisfying (4). If we put

$$u_i = \left(1 + (-1)^i \sqrt{-1}\,J\right)\xi_i \qquad (i = 1, \cdots, k)$$

then it is easily seen that $u_i \neq 0$ and the subspace

$$V_1 = \mathbb{C}u_1 \oplus \mathbb{C}u_2 \oplus \cdots \oplus \mathbb{C}u_k$$

is a J-invariant and U-irreducible subspace of V.

(ii) Assume that the action of U on $\mathbb{C}^{2n}$ is irreducible. Here we use the following basis of $\mathfrak{u}^{\mathbb{C}}$

$$H = -\sqrt{-1}X_2,\ X = \frac{1}{2}(X_3 + \sqrt{-1}X_1),\ Y = \frac{1}{2}(-X_3 + \sqrt{-1}X_1) = \overline{{}^tX}.$$

Let g be an element of $K = U(n)$ such that $\mathrm{Ad}(g)H$ is contained in the maximal torus of $\mathfrak{p}$, namely

$$\mathrm{Ad}(g)H = \sum_{i=1}^{n} \lambda_i\, E_{i,i} - \sum_{i=1}^{n} \lambda_i\, E_{n+i,n+i} \quad (\lambda_1 \geq \cdots \lambda_n \geq 0).$$

By Theorem 2.2, there exist complex numbers γ_i with

$$\begin{aligned}\mathrm{Ad}(g)X &= \sum_{i=1}^{n-1} \gamma_i\, E_{i,i+1} + \gamma_n\, E_{n,2n} + \sum_{i=1}^{n-1} \gamma_{2n-i}\, E_{n+i+1,n+i} \\ \mathrm{Ad}(g)Y &= \sum_{i=1}^{n-1} \overline{\gamma}_i\, E_{i+1,i} + \overline{\gamma}_n\, E_{2n,n} + \sum_{i=1}^{n-1} \overline{\gamma}_{2n-i}\, E_{n+i,n+1+i}\end{aligned}$$

where $|\gamma_i| = \sqrt{i(2n-i)}$ $(1 \leq i \leq 2n-1)$. If we denote by ξ_i the i-th column vector of g^{-1}, from $X \circ J = J \circ Y$, we have

$$\begin{aligned}X(J\,\xi_i) &= \gamma_{2n-i}\,\xi_{n+i+1} = J(Y \cdot \xi_i) = J(\overline{\gamma_i}\xi_{i+1}) = \overline{\gamma_i}\,\xi_{n+i+1} \qquad (1 \leq i \leq n-1) \\ X(J\,\xi_n) &= \gamma_n\,\xi_n = J(Y \cdot \xi_n) = J(\overline{\gamma_n}\xi_{2n}) = -\overline{\gamma_n}\,\xi_n.\end{aligned}$$

Hence we have

$$\gamma_{2n-i} = \overline{\gamma_i} \quad (1 \le i \le n-1), \qquad \gamma_n = \pm\sqrt{-1}\, n.$$

An element g' of $U(n)$ satisfying $\mathrm{Ad}(g')\mathrm{Ad}(g)H = \mathrm{Ad}(g)H$ is of the following form

$$g' = \mathrm{diag}(\varepsilon_1, \cdots, \varepsilon_n, \varepsilon_1, \cdots, \varepsilon_n), \quad |\varepsilon_1| = \cdots = |\varepsilon_n| = 1.$$

If $g' = \mathrm{diag}(\varepsilon_1, \cdots, \varepsilon_n, \varepsilon_1, \cdots, \varepsilon_n)$ $\quad (|\varepsilon_1| = \cdots = |\varepsilon_n| = 1)$ then we have

$$\mathrm{Ad}(g')\mathrm{Ad}(g)X = \sum_{i=1}^{n-1}(\overline{\varepsilon_{i+1}}\varepsilon_i)\gamma_i\, E_{i,i+1} + (\overline{\varepsilon_n}\varepsilon_n)\gamma_n\, E_{n,2n} + \sum_{i=1}^{n-1}\overline{(\overline{\varepsilon_{i+1}}\varepsilon_i)\gamma_i}\, E_{n+i+1,n+i}.$$

Since the sign of the coefficient γ_n of $E_{n,2n}$ does not change under the action of $\mathrm{Ad}(g')$, Lie triple system $\mathbb{R}\widetilde{X}_2 + \mathbb{R}\widetilde{X}_{3,1}$ and $\mathbb{R}\widetilde{X}_2 + \mathbb{R}\widetilde{X}_{3,-1}$ are not congruent under the action of $U(n)$.

Let ε_i $(1 \le i \le n-1)$ be unit complex numbers with

$$\arg\varepsilon_i - \arg\varepsilon_{i+1} + \arg\gamma_i = \frac{\pi}{2} \quad (1 \le i \le n-1).$$

Then the nonzero coefficients $\overline{\varepsilon_{i+1}}\,\varepsilon_i\,\gamma_i$ $(1 \le i \le n-1)$ of $\mathrm{Ad}(g')\mathrm{Ad}(g)X$ satisfy $-\sqrt{-1}\,\overline{\varepsilon_{i+1}}\,\varepsilon_i\,\gamma_i > 0$. Namely we have

$$\varepsilon_{i+1}\,\overline{\varepsilon_i}\,\gamma_i = \sqrt{-1}\,\sqrt{i(2n-1)}$$

and

$$\mathrm{Ad}(g')\mathrm{Ad}(g)X = \sqrt{-1}\left[\sum_{i=1}^{n-1}\sqrt{i(2n-i)}\,\left(E_{i,i+1} + E_{n+i+1,n+i}\right) \pm n\,E_{n,2n}\right].$$

From $X_2 = \sqrt{-1}\,H$ and $X_3 = X - {}^t\overline{X}$ we obtain (15) and (16). □

EXAMPLE 5. The subspace spanned by the following matrices is a Lie triple system of $Sp(3)/U(3)$ $(\varepsilon = \pm 1)$

$$X_2 = \sqrt{-1}\left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right],$$

$$X_3 = \sqrt{-1}\left[\begin{array}{ccc|ccc} 0 & \sqrt{5} & 0 & 0 & 0 & 0 \\ \sqrt{5} & 0 & 2\sqrt{2} & 0 & 0 & 0 \\ 0 & 2\sqrt{2} & 0 & 0 & 0 & \varepsilon \\ \hline 0 & 0 & 0 & 0 & -\sqrt{5} & 0 \\ 0 & 0 & 0 & -\sqrt{5} & 0 & -2\sqrt{2} \\ 0 & 0 & \varepsilon & 0 & -2\sqrt{2} & 0 \end{array}\right].$$

Lie triple system corresponding to $\varepsilon = 1$ and that to $\varepsilon = -1$ are not congruent under the action of $U(3)$.

3.5. Type CII : $Sp(p+q)/Sp(p)\times Sp(q)$. Let θ be the involutive automorphism on $G = Sp(p+q)$ defined by

$$\theta(g) = K_{p,q} \circ g \circ K_{p,q}$$

and put

$$K = \{g \in Sp(p+q) : \theta(g) = g\} = Sp(p) \times Sp(q).$$

Theorem 3.6. *Let M be a non-flat totally geodesic surface of $Sp(p+q)/Sp(p)\times Sp(q)$, $\mathfrak{m}$ be the corresponding Lie triple system and U be the connected Lie subgroup of $Sp(p+q)$ with Lie algebra $\mathfrak{u} = [\mathfrak{m}, \mathfrak{m}] + \mathfrak{m}$. Take a basis X_2, X_3 of $\mathfrak{m}$ satisfying* (7).

(i) *There exist an orthogonal direct sum decomposition*

$$\mathbb{C}^{2(p+q)} = \mathbb{Q}^{p+q} = V_0 \oplus V_1 \oplus \widetilde{J}V_1 \cdots \oplus V_k \oplus \widetilde{J}V_k$$

where V_0 is a trivial $\mathfrak{u}$-module and V_i $(i = 1, \cdots, k)$ are $K_{p,q}$-invariant and U-irreducible subspaces of $\mathfrak{u}$ with $\dim V_i \geq 2$.

(ii) *Corresponding to the direct sum decomposition in* (i)*, the matrix representations of X_2 and X_3 are decomposed into blocks. Thus, without loss of generality, we assume that $\mathbb{C}^{2(p+q)} = V_1 \oplus \widetilde{J}V_1$ and $\left(V_1, \widetilde{J}V_1\right) = \{0\}$. There exists an element $g = [\xi_1, \cdots, \xi_n, \tilde{J}\xi_1, \cdots, \tilde{J}\xi_n] \in Sp(p)\times Sp(q)$ such that*

$$\begin{aligned}
(17)\qquad \mathrm{Ad}(g)X_2 &= \sqrt{-1}\left[\sum_{i=1}^{\min(p-1,q)} \sqrt{2i(n-2i)}\left(-S_{i+1,p+i} + S_{n+p+i,n+i+1}\right)\right.\\
&\quad \left. + \sum_{i=1}^{\min(p,q)} \sqrt{(2i-1)(n+1-2i)}\left(-S_{p+i,i} + S_{n+i,n+p+i}\right)\right],\\
(18)\qquad \mathrm{Ad}(g)X_3 &= \sum_{i=1}^{\min(p-1,q)} \sqrt{2i(n-2i)}\left(G_{i+1,p+i} - G_{n+p+i,n+i+1}\right)\\
&\quad + \sum_{i=1}^{\min(p,q)} \sqrt{(2i-1)(n+1-2i)}\left(G_{p+i,i} - G_{n+i,n+p+i}\right).
\end{aligned}$$

Proof. Put $X_1 = 1/2\,[X_2, X_3]$ and

$$H = -\sqrt{-1}X_1,\ X = \frac{1}{2}(X_2 - \sqrt{-1}X_3),\ Y = -\frac{1}{2}(X_2 + \sqrt{-1}X_3).$$

From

$$K_{p,q} \circ X_1 = X_1 \circ K_{p,q}, \quad K_{p,q} \circ X_i = -X_i \circ K_{p,q} \quad (i = 2, 3),$$

$$[X_1, X_2] = 2X_3, \quad [X_2, X_3] = 2X_1, \quad [X_3, X_1] = 2X_2.$$

we have

$$K_{p,q} \circ H = H \circ K_{p,q}, \quad K_{p,q} \circ X = -X \circ K_{p,q}, \quad K_{p,q} \circ Y = -Y \circ K_{p,q},$$

$$\tilde{J} \circ H = -H \circ \tilde{J}, \quad \tilde{J} \circ X = -Y \circ \tilde{J}.$$

(i) It suffices to show that, in any $K_{p,q}$-invariant and U-invariant subspace V of $\mathbb{C}^{2(p+q)}$, there exists a $K_{p,q}$-invariant and U-irreducible subspace V_1 with $\left(\tilde{J}V_1, V_1\right) = 0$.

Let V be a $K_{p,q}$-invariant and U-invariant subspace of $\mathbb{C}^{2(p+q)}$ and V' be a U-irreducible subspace of V.

We put $k = \dim V'$ and take an orthonormal basis $\xi_1, \cdots, \xi_k$ of V' satisfying (4). It is easily verified that the subspace

$$V_1 = \bigoplus_{i=1}^{k} \mathbb{C}\,(1 + (-1)^i K_{p,q})\,\xi_i$$

is a $K_{p,q}$-invariant and U-irreducible subspace.

Assume that V_1 satisfy $\tilde{J}V_1 = V_1$. Since $\tilde{J}$ gives a structure map of V_1, the dimension of V_1 is an even integer. Put $\dim V_1 = k = 2k'$.

From $H \circ K_{p,q} = K_{p,q} \circ H$ and $K_{p,q}^2 = -1$, we have $K_{p,q}\,\xi_i = \pm\xi_i$ for each i $(1 \leq i \leq k)$. Without loss of generality, we may assume $K_{p,q}\,\xi_1 = \xi_1$. From $Y \circ K_{p,q} = -K_{p,q} \circ Y$, we have

$$K_{p,q}\,\xi_i = (-1)^i \xi_i \quad (1 \leq i \leq k). \tag{19}$$

From $H \circ \tilde{J} = -\tilde{J} \circ H$, we conclude that there exist unit complex numbers α_i such that

$$\tilde{J}\xi_i = \alpha_i\,\xi_{k+1-i} \quad (1 \leq i \leq k),$$

and from $\tilde{J}^2 = -1$ we have

$$\alpha_i = -\alpha_{k+1-i} \quad (1 \leq i \leq k).$$

From $\alpha_i = \left(\xi_{k+1-i}, \tilde{J}\xi_i\right) = -\left(\tilde{J}\,\xi_{k+1-i}, \xi_i\right) = -\overline{\alpha_{k+1-i}} = \overline{\alpha_i}$, we have $\alpha_i = \pm 1$ for each i $(1 \leq i \leq k)$. By changing signs of $\xi_{k'+1}, \cdots, \xi_k$, if necessary, we may assume

$$\tilde{J}\xi_i = \xi_{k+1-i}, \quad \tilde{J}\xi_{k'+i} = -\xi_{k'+1-i} \quad (1 \leq i \leq k').$$

From $\tilde{J} \circ K_{p,q} = K_{p,q} \circ \tilde{J}$, we have

$$K_{p,q}\tilde{J}\xi_1 = \xi_k = \tilde{J}K_{p,q}\xi_1 = -\xi_k,$$

which is a contradiction.

Thus, for each $K_{p,q}$-invariant and U-irreducible subspace V_1 of V, we have $\tilde{J}V_1 \cap V_1 = \{0\}$. Furthermore, by Proposition 2.4, we have $\left(\tilde{J}V_1, V_1\right) = 0$.

(ii) We assume that

$$\mathbb{C}^{2(p+q)} = V_1 \oplus \tilde{J}V_1, \quad \left(\tilde{J}V_1, V_1\right) = \{0\}.$$

We denote by n the dimension of V_1 and take an orthonormal basis $\xi_1, \cdots, \xi_n$ of V_1 satisfying (4). Then there exists an element $g \in Sp(p) \times Sp(q)$ such that

$$\begin{array}{lclclcll} g^{-1}(e_i) & = & \xi_{2i-1}, & g^{-1}(e_{n+i}) & = & \tilde{J}\xi_{2i-1} & (1 \leq i \leq p), \\ g^{-1}(e_{p+i}) & = & \xi_{2i}, & g^{-1}(e_{n+p+i}) & = & \tilde{J}\xi_{2i} & (1 \leq i \leq q). \end{array}$$

By (4), we have

$$\begin{cases} (Ad(g)Y)(e_i) & = & \sqrt{(2i-1)(n+1-2i)}\,e_{p+i} & (1 \leq i \leq \min(p,q)) \\ (Ad(g)Y)(e_{p+i}) & = & \sqrt{2i(n-2i)}\,e_{i+1} & (1 \leq i \leq \min(p-1,q)) \\ (Ad(g)Y)(e_{n+i}) & = & -\sqrt{2i(n-2i)}\,e_{n+p+i} & (1 \leq i \leq \min(p-1,q)) \\ (Ad(g)Y)(e_{n+p+i}) & = & -\sqrt{(2i-1)(n+1-2i)}\,e_{n+i} & (1 \leq i \leq \min(p,q)) \end{cases}$$

From $X_2 = {}^t\overline{Y} - Y$ and $X_3 = \sqrt{-1}\,({}^t\overline{Y} + Y)$ we obtain (17) and (18). □

EXAMPLE 6. (1) The subspace spanned by the following matrices is a Lie triple system of $Sp(3)/Sp(2) \times Sp(1)$

$$X_2 = \sqrt{-1}\left[\begin{array}{cc:c|cc:c} 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ \hdashline \sqrt{2} & \sqrt{2} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -\sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{2} \\ \hdashline 0 & 0 & 0 & -\sqrt{2} & -\sqrt{2} & 0 \end{array}\right],$$

$$X_3 = \left[\begin{array}{cc:c|cc:c} 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2} & 0 & 0 & 0 \\ \hdashline -\sqrt{2} & \sqrt{2} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{2} \\ \hdashline 0 & 0 & 0 & -\sqrt{2} & \sqrt{2} & 0 \end{array}\right],$$

(2) The subspace spanned by the following matrices is a Lie triple system of $Sp(4)/Sp(2)\times Sp(2)$

$$X_2 = \sqrt{-1}\left[\begin{array}{cc:cc|cc:cc} 0 & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & \sqrt{3} & 0 & 0 & 0 & 0 \\ \hdashline \sqrt{3} & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & -\sqrt{3} \\ \hdashline 0 & 0 & 0 & 0 & -\sqrt{3} & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 \end{array}\right],$$

$$X_3 = \left[\begin{array}{cc:cc|cc:cc} 0 & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & \sqrt{3} & 0 & 0 & 0 & 0 \\ \hdashline -\sqrt{3} & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & \sqrt{3} \\ \hdashline 0 & 0 & 0 & 0 & -\sqrt{3} & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 \end{array}\right].$$

3.6. Type BDI : $SO(p+q)/S(O(p)\times O(q))$. Let θ be the involutive automorphism on $G = SO(p+q)$ defined by

$$\theta(g) = I_{p,q} \circ g \circ I_{p,q}$$

and put

$$K = \{g \in SO(p+q) : \theta(g) = g\} = S(O(p) \times O(q)).$$

Theorem 3.7. *Let M be a non-flat totally geodesic surface of $SO(p+q)/S(O(p)\times O(q))$, $\mathfrak{m}$ be the corresponding Lie triple system and U be the connected Lie subgroup of $SO(p+q)$ with Lie algebra $\mathfrak{u} = [\mathfrak{m},\mathfrak{m}] + \mathfrak{m}$. Take a basis X_2, X_3 of $\mathfrak{m}$ satisfying* (7)*.*

(i) *There exists an orthogonal direct sum decomposition*

$$\mathbb{R}^{p+q} = V_0 \oplus V_1 \oplus \cdots \oplus V_k,$$

where V_0 is a trivial $\mathfrak{u}$-module and V_i $(i = 1,\cdots,k)$ are $I_{p,q}$-invariant and $\mathfrak{u}$-irreducible subspaces with $\dim V_i \geq 3$.

(ii) *For each $I_{p,q}$-invariant, U-irreducible subspace V of $\mathbb{R}^{p+q}$, we have*

$$\left|\dim\{v \in V : I_{p,q}(v) = v\} - \dim\{v \in V : I_{p,q}(v) = -v\}\right| \leq 1.$$

(iii) *Corresponding to the direct sum decomposition in* (i)*, the matrix representations of X_2 and X_3 are decomposed into blocks. Thus, without loss of generality, we assume that that the action of U on $\mathbb{R}^{p+q}$ is irreducible.*

Case 1: $p = q+1 \geq 2$. We denote by p' the integer part of $p/2$ and by q' the integer part of $q/2$. There exists an element $g \in S(O(p)\times O(q))$ such that

$$\begin{aligned}
(20)\qquad \mathrm{Ad}(g)X_2 &= -\sum_{i=1}^{q'} \sqrt{(2i-1)(p+q+1-2i)}\left(G_{p+2i-1,2i-1} + G_{p+2i,2i}\right)\\
&\quad + \sum_{i=1}^{p'-1} \sqrt{2i(p+q-2i)}\left(G_{p+2i-1,2i+1} + G_{p+2i,2i+2}\right)\\
&\quad + \begin{cases} -\sqrt{2}\sqrt{p\,q}G_{p+q,q} & (\text{if } p = 0 \pmod 2)\\ \sqrt{2}\sqrt{p\,q}G_{p+q-1,p} & (\text{if } p = 1 \pmod 2),\end{cases}
\end{aligned}$$

$$\begin{aligned}
(21)\qquad \mathrm{Ad}(g)X_3 &= \sum_{i=1}^{q'} \sqrt{(2i-1)(p+q+1-2i)}\left(G_{p+2i,2i-1} - G_{p+2i-1,2i}\right)\\
&\quad + \sum_{i=1}^{p'-1} \sqrt{2i(p+q-2i)}\left(G_{p+2i,2i+1} - G_{p+2i-1,2i+2}\right)\\
&\quad + \begin{cases} -\sqrt{2}\sqrt{p\,q}G_{p+q,p} & (\text{if } p = 0 \pmod 2)\\ \sqrt{2}\sqrt{p\,q}G_{p+q,p} & (\text{if } p = 1 \pmod 2).\end{cases}
\end{aligned}$$

Case 2: $p = q$. p is an even integer, say $p = 2p'$, and there exists an element $g \in S(O(p)\times O(q))$ such that

$$(22)\qquad \mathrm{Ad}(g)X_2 = \sum_{i=1}^{p'-1}\sqrt{2i(p-2i)}\left(G_{p+i,i+1}+G_{p+p'+i,p'+i+1}\right) - \sum_{i=1}^{p'}\sqrt{(2i-1)(p+1-2i)}\left(G_{p+i,i}+G_{p+p'+i,p'+i}\right),$$

$$(23)\qquad \mathrm{Ad}(g)X_3 = \sum_{i=1}^{p'-1}\sqrt{2i(p-2i)}\left(G_{p+p'+i,i+1}-G_{p+i,p'+i+1}\right) + \sum_{i=1}^{p'}\sqrt{(2i-1)(p+1-2i)}\left(G_{p+p'+i,i}-G_{p+i,p'+i}\right).$$

Proof. Put $X_1 = 1/2\,[X_2, X_3]$ and

$$H = -\sqrt{-1}\,X_1, \quad X = \frac{1}{2}\left(X_2 - \sqrt{-1}\,X_3\right), \quad Y = -\frac{1}{2}\left(X_2 + \sqrt{-1}\,X_3\right).$$

We have

$$I_{p,q}\circ X_1 = X_1\circ I_{p,q}, \quad I_{p,q}\circ X_i = -X_i\circ I_{p,q} \quad (i = 2,\ 3),$$
$$[X_1, X_2] = 2X_3, \quad [X_2, X_3] = 2X_1, \quad [X_3, X_1] = 2X_2.$$

(i) It suffices to show that, in any $I_{p,q}$-invariant and U-invariant subspace V of $\mathbb{R}^{p+q}$, there exists an $I_{p,q}$-invariant and U-irreducible subspace V_1.

Let V be an $I_{p,q}$-invariant and U-invariant subspace of $\mathbb{R}^{p+q}$ and V' be a U-irreducible subspace of V.

Case 1: Assume that $V'^{\mathbb{C}}$ is irreducible over $\mathbb{C}$.

In this case $\dim V'$ is an odd integer. If V' is a τ-invariant subspace then its real part $\{v \in V' : \tau(v) = v\}$ is an $I_{p,q}$-invariant and U-irreducible subspace of V.

Assume that $\tau(V') \cap V' = \{0\}$. Put $\dim V' = k$ and take an orthonormal basis $\xi_1, \xi_2, \cdots,$ ξ_k of $V'^{\mathbb{C}}$ which satisfy (4). From $\tau\circ H = -H\circ\tau$, there exist complex numbers α_i with

$$\tau\,\xi_i = \alpha_i\,\xi_{k+1-i}, \quad |\alpha_i| = 1 \quad (1 \le i \le k).$$

If we put

$$u_i = \left(1 + (-1)^i\, I_{p,q}\right)\xi_i,$$

it is easily verified that the subspace

$$V'' = \mathbb{C}u_1 \oplus\cdots\oplus \mathbb{C}u_k \subset W^{\mathbb{C}}$$

is an $I_{p,q}$-invariant, τ-invariant and U-irreducible subspace over $\mathbb{C}$ and its real part $V_1 = \{v \in V'' : \tau v = v\}$ is an $I_{p,q}$-invariant and U-irreducible subspace of V.

Case 2: Assume that $V'^{\mathbb{C}}$ is reducible over $\mathbb{C}$.

In this case, by Lemma 2.3, $\dim_{\mathbb{R}} V'$ is an integer divided by 4 and, by [1, p.27], there exists an irreducible subspace V'' of $V'^{\mathbb{C}}$ such that

$$V'^{\mathbb{C}} = V'' \oplus \tau(V'').$$

If we put $E_1 = \{x \in V'' \oplus \tau(V'') : H\cdot x = (n-1)x\}$, then E_1 is an $I_{p,q}$-invariant subspace.

Let $\xi_1 \in E_1$ be an eigenvector of $I_{p,q}$. Assume that $I_{p,q}\xi_1 = \xi_1$. If we put

$$\xi_i = \frac{1}{\Pi_{j=1}^{i-1}\sqrt{j(n-j)}} Y^{i-1}\,\xi_1 \quad (2 \le i \le k), \quad \eta_i = \tau(\xi_i) \quad (1 \le i \le k), \tag{24}$$

we can easily verify

$$\begin{cases} H\cdot\xi_i &= (k+1-2i)\,\xi_i, & H\cdot\eta_i &= -(k+1-2i)\,\eta_i,\\ X\cdot\xi_i &= \gamma_{i-1}\,\xi_{i-1}, & X\cdot\eta_i &= -\gamma_i\,\eta_{i+1},\\ Y\cdot\xi_i &= \gamma_i\,\xi_{i+1}, & Y\cdot\eta_i &= -\gamma_{i-1}\,\eta_{i-1},\\ I_{p,q}\xi_i &= (-1)^{i-1}\,\xi_i, & I_{p,q}\eta_i &= (-1)^{i-1}\,\eta_i \end{cases} \tag{25}$$

for $1 \le i \le k$, where we put $\gamma_i = \sqrt{i(k-i)}$ and $\xi_0 = \xi_{n+1} = \eta_0 = \eta_{n+1} = 0$. Thus the subspaces

$$W = \oplus_{i=1}^{n}\mathbb{C}\,\xi_i, \quad \tau(W) = \oplus_{i=1}^{n}\mathbb{C}\,\eta_i,$$

are $I_{p,q}$invariant and U-irreducible subspaces of $V^{\mathbb{C}}$ and the real part

$$V_1 = \{x \in W \oplus \tau(W) : \tau x = x\}$$

is an $I_{p,q}$-invariant and U-irreducible subspace of V.

(ii) If M is a non-flat totally geodesic surface of $SO(p+q)/SO(p)\times SO(q)$ $(p \ge q)$, then by the totally geodesic embedding $SO(p+q)/S(O(p)\times O(q)) \to SU(p+q)/S(U(p)\times U(q))$ M is also a totally geodesic surface of $SU(p+q)/S(U(p)\times U(q))$. Thus we have $|p-q| \le 1$.

(iii) We assume that the action on $\mathbb{R}^{p+q}$ is irreducible. From (ii) there are two possible cases; $p = q+1$ and $p = q$.

Case 1: $p = q+1$. Let $g \in \mathrm{Ad}(S(O(p)\times O(q)))$ be an element such that $\mathrm{Ad}(g)H$ is contained in the maximal torus of $\mathfrak{k}$, namely $\mathrm{Ad}(g)X_1$ is of the form

$$\mathrm{Ad}(g)X_1 = -\sqrt{-1}\left(\sum_{i=1}^{p'} a_i\,G_{2i-1,2i} + \sum_{i=1}^{q'} b_i\,G_{p+2i-1,p+2i}\right)$$

where $a_1 \ge \cdots \ge a_{p'} > 0$ and $b_1 \ge \cdots \ge b_{q'} > 0$.

We denote by g_i the i-th column vector of g^{-1} and put

$$\begin{cases} u_{2i-1} &= \frac{1}{\sqrt{2}}\left(g_{2i-1} + \sqrt{-1}g_{2i}\right) & (1 \le i \le p')\\ u_{p+q+2-2i} &= \frac{1}{\sqrt{2}}\left(g_{2i-1} - \sqrt{-1}g_{2i}\right) & (1 \le i \le p')\\ u_{2i} &= \frac{1}{\sqrt{2}}\left(g_{p+2i-1} + \sqrt{-1}g_{p+2i}\right) & (1 \le i \le q')\\ u_{p+q+1-2i} &= \frac{1}{\sqrt{2}}\left(g_{p+2i-1} - \sqrt{-1}g_{p+2i}\right) & (1 \le i \le q')\\ u_p &= \begin{cases} g_p & (\text{if } q = 0 \pmod 2)\\ g_{p+q} & (\text{if } q = 1 \pmod 2)\end{cases} \end{cases}$$

so that

$$\begin{cases} X_1 \cdot u_{2i-1} &= a_i\, u_{2i-1} & (1 \leq i \leq p'), \\ X_1 \cdot u_{p+q+2-2i} &= -a_i\, u_{p+q+2-2i} & (1 \leq i \leq p'), \\ X_1 \cdot u_{2i} &= b_i\, u_{2i} & (1 \leq i \leq q'), \\ X_1 \cdot u_{p+q+1-2i} &= -b_i\, u_{p+q+1-2i} & (1 \leq i \leq q'), \\ X_1 \cdot u_p &= 0 \end{cases}$$

hold.

By Theorem 2.2, we have $\{a_1, \cdots, a_{p'}, b_1, \cdots, b_{q'}\} = \{p+q-1, p+q-3, \cdots, 1\}$ and there exist complex numbers γ_i such that

$$\begin{array}{llll} Xu_2 = \gamma_1\, u_1, & Xu_3 = \gamma_2\, u_2, & \cdots, & Xu_{p+q} = \gamma_{p+q-1}\, u_{p+q-1}, \\ Yu_1 = \overline{\gamma_1}\, u_2, & Yu_2 = \overline{\gamma_2}\, u_3, & \cdots, & Yu_{p+q-1} = \overline{\gamma_{p+q-1}}\, u_{p+q}, \end{array}$$

where $|\gamma_i| = \sqrt{i(p+q-i)}$ $(1 \leq i \leq p+q-1)$. From $\tau \circ Y = -X \circ \tau$, we have

$$\tau(Yu_i) = \tau(\overline{\gamma_i}\, u_{i+1}) = \gamma_i\, u_{p+q-i} = -X(\tau u_i) = -Xu_{p+q+1-i} = -\gamma_{p+q-i} u_{p+q-i}$$

for $1 \leq i \leq p' + q' = q$. Thus we have

$$\gamma_i = -\gamma_{p+q-i} \quad (1 \leq i \leq q).$$

Let h be an element of $S(O(p) \times O(q))$ such that the restriction of h on $\mathbb{R}g_i \oplus \mathbb{R}g_{p+q+1-i}$ is the rotation through an angle θ_i $(1 \leq i \leq q)$ and $h(u_p) = u_p$. Then we have

$$h(u_i) = e^{\sqrt{-1}\theta_i}\, u_i, \; h(u_{p+q+1-i}) = e^{-\sqrt{-1}\theta_i}\, u_{p+q+1-i} \quad (1 \leq i \leq q).$$

If we put $u'_i = h(u_i)$ then

$$X \cdot u'_i = e^{\sqrt{-1}(\theta_i - \theta_{i-1})} \gamma_{i-1}\, u'_{i-1}, \quad X \cdot u'_p = e^{\sqrt{-1}\theta_p} \gamma_{p-1}\, u'_{p-1}$$

hold. We can choose θ_i so that

$$X \cdot u'_{i+1} = \sqrt{i(p+q-i)} u'_i \quad (1 \leq i \leq q).$$

Now we put

$$\gamma_i = -\gamma_{p+q-i} = \sqrt{i(p+q-i)} \quad (0 \leq i \leq q).$$

From

$$\begin{cases} g_{2i-1} &= \frac{\sqrt{2}}{2}\left(u_{2i-1} + u_{p+q+2-2i}\right) & (1 \leq i \leq p') \\ g_{2i} &= -\sqrt{-1}\, \frac{\sqrt{2}}{2}\left(u_{2i-1} - u_{p+q+2-2i}\right) & (1 \leq i \leq p') \\ g_{p+2i-1} &= \frac{\sqrt{2}}{2}\left(u_{2i} + u_{p+q+1-2i}\right) & (1 \leq i \leq q') \\ g_{p+2i} &= -\sqrt{-1}\, \frac{\sqrt{2}}{2}\left(u_{2i} - u_{p+q+1-2i}\right) & (1 \leq i \leq q') \\ g_p &= u_p & (\text{if } p = 1 \pmod 2) \\ g_{p+q} &= u_p & (\text{if } p = 0 \pmod 2), \end{cases}$$

we have

$$\begin{aligned} X_2 \cdot g_{2i-1} &= \frac{\sqrt{2}}{2} (X - Y)(u_{2i-1} + u_{p+q+2-2i}) \\ &= \frac{\sqrt{2}}{2} \left(\gamma_{2i-2}\, u_{2i-2} + \gamma_{p+q+1-2i} u_{p+q+1-2i} - \gamma_{2i-1} u_{2i} - \gamma_{p+q+2-2i} u_{p+q+3-2i}\right) \end{aligned}$$

$$= \begin{cases} -\gamma_{2i-1}\, g_{p+2i-1} + \gamma_{2i-2}\, g_{p+2i-3} & (\text{if } 1 \le i \le q') \\ -\sqrt{2}\gamma_{p-1}\, g_{p+q} + \gamma_{p-2}\, g_{2p-3} & (\text{if } i = p' = q' + 1), \end{cases}$$

$$\begin{aligned} X_2 \cdot g_{2i} &= -\sqrt{-1}\,\frac{\sqrt{2}}{2}\,(X - Y)(u_{2i-1} - u_{p+q+2-2i}) \\ &= -\sqrt{-1}\,\frac{\sqrt{2}}{2}\left(\gamma_{2i-2}\, u_{2i-2} - \gamma_{p+q+1-2i} u_{p+q+1-2i} - \gamma_{2i-1} u_{2i} + \gamma_{p+q+2-2i} u_{p+q+3-2i}\right) \\ &= \begin{cases} \gamma_{2i-2}\, g_{p+2i-2} - \gamma_{2i-1}\, g_{p+2i} & (\text{if } 1 \le i \le q') \\ \gamma_{p-2}\, g_{2p-2} & (\text{if } i = p' = q' + 1), \end{cases} \end{aligned}$$

$$\begin{aligned} X_2 \cdot g_{p+2i-1} &= \frac{\sqrt{2}}{2}\,(X - Y)(u_{2i} + u_{p+q+1-2i}) \\ &= \frac{\sqrt{2}}{2}\left(\gamma_{2i-1}\, u_{2i-1} + \gamma_{p+q-2i} u_{p+q-2i} - \gamma_{2i} u_{2i+1} - \gamma_{p+q+1-2i} u_{p+q+2-2i}\right) \\ &= \begin{cases} \gamma_{2i-1}\, g_{2i-1} - \gamma_{2i}\, g_{2i+1} & (\text{if } 1 \le i < q' \text{ or } i = q' < p') \\ \gamma_{q-1}\, g_{q-1} - \sqrt{2}\gamma_q g_p & (\text{if } i = p' = q'), \end{cases} \end{aligned}$$

$$\begin{aligned} X_2 \cdot g_{p+2i} &= -\sqrt{-1}\frac{\sqrt{2}}{2}\,(X - Y)(u_{2i} - u_{p+q+1-2i}) \\ &= -\sqrt{-1}\frac{\sqrt{2}}{2}\left(\gamma_{2i-1}\, u_{2i-1} - \gamma_{p+q-2i} u_{p+q-2i} - \gamma_{2i} u_{2i+1} + \gamma_{p+q+1-2i} u_{p+q+2-2i}\right) \\ &= \begin{cases} \gamma_{2i-1}\, g_{2i} - \gamma_{2i}\, g_{2i+2} & (\text{if } 1 \le i \le p' - 1) \\ \gamma_{q-1}\, g_q & (\text{if } i = p' = q'), \end{cases} \end{aligned}$$

and

$$\begin{cases} X_2 \cdot g_p = (X - Y)u_p = \sqrt{2}\,\gamma_{p-1}\, g_{p+q-1} & (\text{if } p = 1 \pmod 2) \\ X_2 \cdot g_{p+q} = (X - Y)u_{p+q} = \sqrt{2}\,\gamma_{p-1}\, g_{p-1} & (\text{if } p = 0 \pmod 2). \end{cases}$$

Thus we obtain (20). The formula (21) is obtained similarly.

Case 2: $p = q$

Since, $p + q$ is divided by 4 from Lemma 2.3, we have $p = q \equiv 0 \pmod 2$.

We use the basis $\xi_1,\cdots,\xi_p,\eta_1,\cdots,\eta_p$ defined in (24) (replacing k by p). Put $p = 2p'$. The set of vectors

$$\begin{aligned} u_i &= \tfrac{\sqrt{2}}{2}\,(\xi_{2i-1} + \eta_{2i-1})\,, & u_{p'+i} &= \sqrt{-1}\,\tfrac{\sqrt{2}}{2}\,(\xi_{2i-1} - \eta_{2i-1})\,, \\ u_{p+i} &= \tfrac{\sqrt{2}}{2}\,(\xi_{2i} + \eta_{2i})\,, & e_{p+p'+i} &= \sqrt{-1}\,\tfrac{\sqrt{2}}{2}\,(\xi_{2i} - \eta_{2i})\,, \end{aligned}$$

where i runs through $1 \le i \le p'$, forms an orthonormal basis of $\mathbb{R}^{p+q}$ with

$$I_{p,q}\, u_i = u_i, \quad I_{p,q}\, u_{p+i} = -u_{p+i}, \quad (1 \le i \le p).$$

From (25), for $1 \le i \le p'$, we have

$$X_2 \cdot u_i = \frac{\sqrt{2}}{2}(X - Y) \cdot (\xi_{2i-1} + \eta_{2i-1})$$

$$
\begin{aligned}
&= \frac{\sqrt{2}}{2}\,(\gamma_{2i-2}\xi_{2i-2} - \gamma_{2i-1}\xi_{2i} - \gamma_{2i-1}\eta_{2i} + \gamma_{2i-2}\eta_{2i-2}) \\
&= \begin{cases} -\gamma_1\, u_{p+1} & (\text{if } i = 1) \\ \gamma_{2i-2}\, u_{p+i-1} - \gamma_{2i-1}\, u_{p+i} & (\text{if } 1 < i \leq p'), \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
X_2 \cdot u_{p'+i} &= \sqrt{-1}\frac{\sqrt{2}}{2}(X - Y)\cdot(\xi_{2i-1} - \eta_{2i-1}) \\
&= \sqrt{-1}\frac{\sqrt{2}}{2}\,(\gamma_{2i-2}\xi_{2i-2} - \gamma_{2i-1}\xi_{2i} + \gamma_{2i-1}\eta_{2i} - \gamma_{2i-2}\eta_{2i-2}) \\
&= \begin{cases} -\gamma_1\, u_{p+p'+1} & (\text{if } i = 1) \\ \gamma_{2i-2}\, u_{p+p'+i-1} - \gamma_{2i-1}\, u_{p+p'+i} & (\text{if } 1 < i \leq p'), \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
X_2 \cdot u_{p+i} &= \frac{\sqrt{2}}{2}(X - Y)\cdot(\xi_{2i} + \eta_{2i}) \\
&= \frac{\sqrt{2}}{2}\,(\gamma_{2i-1}\xi_{2i-1} - \gamma_{2i}\xi_{2i+1} - \gamma_{2i}\eta_{2i+1} + \gamma_{2i-1}\eta_{2i-1}) \\
&= \begin{cases} \gamma_{2i-1}\, u_i - \gamma_{2i}\, u_{i+1} & (\text{if } 1 \leq i < p') \\ \gamma_{p-1}\, u_{p'} & (\text{if } i = p'), \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
X_2 \cdot u_{p+p'+i} &= \sqrt{-1}\frac{\sqrt{2}}{2}(X - Y)\cdot(\xi_{2i} - \eta_{2i}) \\
&= \sqrt{-1}\frac{\sqrt{2}}{2}\,(\gamma_{2i-1}\xi_{2i-1} - \gamma_{2i}\xi_{2i+1} + \gamma_{2i}\eta_{2i+1} - \gamma_{2i-1}\eta_{2i-1}) \\
&= \begin{cases} \gamma_{2i-1}\, u_{p'+i} - \gamma_{2i}\, u_{p'+i+1} & (\text{if } 1 \leq i < p') \\ \gamma_{p-1}\, u_p & (\text{if } i = p'). \end{cases}
\end{aligned}
$$

Thus we obtain (22). The formula (23) is obtained similarly. □

Example 7. The subspace spanned by the following vectors is a Lie triple system of $SO(5)/S(O(3)\times O(2))$

$$
X_2 = \left[\begin{array}{ccc:cc} 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -2\sqrt{3} & 0 \\ \hdashline -2 & 0 & 2\sqrt{3} & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \end{array}\right], \quad
X_3 = \left[\begin{array}{ccc:cc} 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -2\sqrt{3} \\ \hdashline 0 & -2 & 0 & 0 & 0 \\ 2 & 0 & 2\sqrt{3} & 0 & 0 \end{array}\right].
$$

Example 8. The subspace spanned by the following vectors is a Lie triple system of $SO(8)/S(O(4)\times O(4))$

$$X_2 = \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & \sqrt{3} \\ \hline -\sqrt{3} & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{3} & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 \end{array}\right],$$

$$X_3 = \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & -\sqrt{3} \\ 0 & 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & \sqrt{3} & 0 & 0 \\ \hline 0 & 0 & -\sqrt{3} & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 \\ \sqrt{3} & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

3.7. Type DIII : $SO(2n)/U(n)$. Let θ be the involutive automorphism on $G = SO(2n)$ defined by

$$\theta(g) = J \circ g \circ J^{-1}$$

and put

$$K = \{g \in SO(2n) : \theta(g) = g\} = U(n).$$

Theorem 3.8. *Let M be a non-flat totally geodesic surface of $SO(2n)/U(n)$, $\mathfrak{m}$ be the corresponding Lie triple system and U be the connected Lie subgroup of $SO(2n)$ with Lie algebra $\mathfrak{u} = [\mathfrak{m}, \mathfrak{m}] + \mathfrak{m}$. Take a basis X_2, X_3 of $\mathfrak{m}$ satisfying* (7).

(i) *There exist an orthogonal direct sum decomposition*

$$\mathbb{R}^{2n} = V_0 \oplus V_1 \oplus \cdots \oplus V_k,$$

where V_0 is a trivial $\mathfrak{u}$-module and V_i $(i = 1, \cdots, k)$ are τ-invariant and $\mathfrak{u}$-irreducible subspaces with $\dim V_i \geq 2$.

(ii) *Corresponding to the direct sum decomposition in* (i), *the matrix representations of X_2 and X_3 are decomposed into blocks. Thus, without loss of genality, we assume that $\mathbb{C}^n = V_1$. There exists an element $g = [u_1, \cdots, u_n, u_{n+1}, \cdots, u_{2n}] \in U(n)$, where $u_{n+1} = J u_1$, $\cdots$, $u_{2n} = J u_n$, such that*

$$\mathrm{Ad}(g)(X_2) = \sum_{i=1}^{n-1} \sqrt{i(n-i)}\, (G_{i,i+1} - G_{n+i,n+i+1}), \tag{26}$$

$$\text{(27)} \qquad \mathrm{Ad}(g)(X_3) \;=\; \sum_{i=1}^{n-1} (-1)^i \sqrt{i(n-i)}\,(G_{n+i,i+1} - G_{n+i+1,i})\,.$$

Proof. Put $X_1 = 1/2\,[X_2, X_3]$ and

$$H = -\sqrt{-1}X_1,\; X = \frac{1}{2}(X_2 - \sqrt{-1}X_3),\; Y = -\frac{1}{2}(X_2 + \sqrt{-1}X_3).$$

From

$$J \circ X_1 = X_1 \circ J, \quad J \circ X_i = -X_i \circ J \quad (i = 2,\ 3),$$
$$[X_1, X_2] = 2X_3, \quad [X_2, X_3] = 2X_1, \quad [X_3, X_1] = 2X_2.$$

we have

$$J \circ H = H \circ J, \quad J \circ X = -X \circ J, \quad J \circ Y = -Y \circ J,$$
$$\tau \circ H = -H \circ \tau, \quad \tau \circ X = -Y \circ \tau.$$

(i) It suffices to show that, in any J-invariant and U-invariant subspace V of $\mathbb{R}^{2n}$, there exists a J-invariant and U-irreducible subspace V_1 of V.

We shall prove that if we take a J-invariant and U-invariant subspace V of $\mathbb{R}^{2n}$, then there exists a J-invariant, U-irreducible and τ-invariant subspace V' of $V^{\mathbb{C}}$ with $(\tau(V'), V') = 0$. For the real part $\{x \in \tau(V') \oplus V' \ : \tau(x) = x\}$ is a J-invariant and U-irreducible subspace of V.

Let $W = V^{\mathbb{C}}$ be a J-invariant, U-invariant and τ-invariant subspace of $\mathbb{C}^{2n}$ and W' be an irreducible component of W.

Put $k = \dim W'$ and take an orthonormal basis $\xi_1,\cdots,\xi_k$ satisfying (4). If we put

$$u_i = (1 + (-1)^i \sqrt{-1}\,J)\,\xi_i \quad [\text{resp. } v_i = (1 - (-1)^i \sqrt{-1}\,J)\,\xi_i]$$

for $1 \le i \le k$, then it is easily seen that

$$W_1' = \oplus_{i=1}^{k} \mathbb{C}\, u_i \quad [\text{resp. } W_2' = \oplus_{i=1}^{k} \mathbb{C}\, v_i]$$

is a J-invariant and U-irreducible subspace. Either W_1' or W_2' is not $\{0\}$. We take one of W_1' or W_2', which is not $\{0\}$, as V'. The dimension of V' is an even integer and thus V' admits a structure map of index (-1). If $\tau(V') = V'$ then τ is a structure map on V' of index 1, which is a contradiction. Namely we have $\tau(V') \cap V' = \{0\}$. By Proposition 2.4, we have $(\tau(V'), V') = 0$.

(ii) For simplicity, we assume that the action of U on $\mathbb{R}^{2n}$ is irreducible.

Let V be the irreducible component of $\mathbb{C}^{2n}$ in the proof of (i). Note that $\mathbb{C}^{2n} = V \oplus \tau(V)$. Take an orthonormal basis $\xi_1,\cdots,\xi_n$ of V satisfying (4) (replacing k by n). Without loss of generality we may assume that $J\xi_1 = \sqrt{-1}\,\xi_1$. If we put $\eta_i = \tau\xi_i$ $(1 \le i \le n)$, we have

$$\left\{\begin{array}{lcllcl} H \cdot \xi_i &=& (n+1-2i)\,\xi_i, & H \cdot \eta_i &=& -(n+1-2i)\,\eta_i, \\ X \cdot \xi_i &=& \gamma_{i-1}\,\xi_{i-1}, & X \cdot \eta_i &=& -\gamma_i\,\eta_{i+1}, \\ Y \cdot \xi_i &=& \gamma_i\,\xi_{i+1}, & Y \cdot \eta_i &=& -\gamma_{i-1}\,\eta_{i-1}, \\ J \cdot \xi_i &=& (-1)^{i-1}\sqrt{-1}\,\xi_i, & J \cdot \eta_i &=& (-1)^i\sqrt{-1}\,\eta_i, \end{array}\right.$$

where we put $\gamma_i = \sqrt{i(n-i)}$ and $\xi_0 = \xi_{n+1} = \eta_0 = \eta_{n+1} = 0$.

The set of vectors

$$u_i = \frac{\sqrt{2}}{2}\,(\xi_i + \eta_i)\,, \quad u_{n+i} = (-1)^{i-1}\,\frac{\sqrt{-2}}{2}\,(\xi_i - \eta_i)\,, \quad (1 \leq i \leq n),$$

forms an orthonormal basis of $\mathbb{R}^{2n}$ with

$$J\,u_i = u_{n+i}, \quad J\,u_{n+i} = -u_i, \quad (1 \leq i \leq n).$$

Since

$$\begin{aligned}
X_2\,u_i &= \frac{\sqrt{2}}{2}(X - Y)(\xi_i + \eta_i) \\
&= \frac{\sqrt{2}}{2}(\gamma_{i-1}\xi_{i-1} - \gamma_i\xi_{i+1} - \gamma_i\eta_{i+1} + \gamma_{i-1}\eta_{i-1}) \\
&= \gamma_{i-1}\,u_{i-1} - \gamma_i\,u_{i+1}, \\
X_2\,u_{n+i} &= (-1)^{i-1}\frac{\sqrt{-2}}{2}(X - Y)(\xi_i - \eta_i) \\
&= (-1)^{i-1}\frac{\sqrt{-2}}{2}\,(\gamma_{i-1}\xi_{i-1} - \gamma_i\xi_{i+1} + \gamma_i\eta_{i+1} - \gamma_{i-1}\eta_{i-1}) \\
&= -\gamma_{i-1}\,u_{n+i-1} + \gamma_i\,u_{n+i+1}
\end{aligned}$$

hold for each i $(1 \leq i \leq n)$, where we put $u_0 = u_{n+1} = 0$, we have (26). The formula (27) is obtained similarly. □

EXAMPLE 9. The subspace spanned by the following matrices is a Lie triple system of $SO(8)/U(4)$

$$X_2 = \left[\begin{array}{cccc:cccc}
0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\sqrt{3} & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & \sqrt{3} & 0 & 0 & 0 & 0 \\
0 & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 & 0 \\
\hdashline
0 & 0 & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{3} & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & -\sqrt{3} \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0
\end{array}\right],$$

$$X_3 = \left[\begin{array}{cccc:cccc}
0 & 0 & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{3} & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & -\sqrt{3} \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 \\
\hdashline
0 & -\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
\sqrt{3} & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0
\end{array}\right].$$

4. Remarks on the curvature

The following question was posed by the referee: Is there a classical irreducible Riemannian symmetric space in which there exist two totally geodesic surfaces which have the same value of the curvature, but which are not congruent to each other.

Let G, $\mathfrak{g}$, θ, K, $\mathfrak{k}$ and $\mathfrak{p}$ be the same as those in the previous section. Let $(\,,\,)$ be a G-invariant inner product of $\mathfrak{g}$. We denote also by $(\,,\,)$ the G-invariant Riemannian metric on $P = G/K$ which coincides with the restriction of $(\,,\,)$ to $\mathfrak{p}$. We denote by R the curvature tensor of $(P, (\,,\,))$.

Proposition 4.1. *Let M be a non-flat totally geodesic surface of $P = G/K$ and $\mathfrak{m}$ be the corresponding Lie triple system. If we denote by κ the sectional curvature of M, then we have*

$$\kappa = \frac{4}{\|X_2\|^2} = \frac{4}{\|X_3\|^2}$$

where X_2, X_3 is a basis of $\mathfrak{m}$ satisfying (7).

Proof. Put $X_1 = \frac{1}{2}[X_2, X_3]$. From $R(X,Y)Z = -[[X,Y],Z]$ ([3, p.215]), we have

$$\kappa = \frac{\|[X_2, X_3]\|^2}{\|X_2\|^2\|X_2\|^2} = \frac{4\|X_1\|^2}{\|X_2\|^2\|X_2\|^2}.$$

The restriction of $(\,,\,)$ to the Lie algebra $\mathfrak{u} = [\mathfrak{m}, \mathfrak{m}] + \mathfrak{m}$ is $\mathfrak{u}$-invariant. Thus it is proportional to the Killing form of $\mathfrak{u} \cong \mathfrak{su}(2)$. From $[X_1, X_2] = 2X_3$,$[X_2, X_3] = 2X_1$ and $[X_3, X_1] = 2X_2$, we have $\|X_1\| = \|X_2\| = \|X_3\|$. □

Hereafter we consider totally geodesic surfaces of $P = SU(n)/SO(n)$.

Define the $SU(n)$-invariant inner product on $\mathfrak{su}(n)$ by

$$(X, Y) = -\mathrm{trace}\, X\, Y \quad (X, Y \in \mathfrak{su}(n)).$$

We denote by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{so}(n)$ in $\mathfrak{su}(n)$ and extend the restriction of $(\,,\,)$ to $\mathfrak{p} = (\mathfrak{so}(n))^{\perp}$ to the $SU(n)$-invariant Riemannian metric on $P = SU(n)/SO(n)$. The maximum of the sectional curvature of P is equal to 2. Before giving examples of pair of totally geodesic surfaces which are not congruent but with the same value of curvature, we mention about the topology of totally geodesic surfaces of $SU(n)/SO(n)$.

Proposition 4.2. *Let M be a non-flat totally geodesic surface of $SU(n)/SO(n)$, $\mathfrak{m}$ be the corresponding Lie triple system and U be the connected Lie subgroup of $SU(n)$ with Lie algebra* $\mathfrak{u} = [\mathfrak{m}, \mathfrak{m}] + \mathfrak{m}$. *Let*

$$\mathbb{C}^n = V_1 \oplus \cdots \oplus V_k$$

be the decomposition of $\mathbb{C}^n$ by τ-invariant and U-irreducible subspaces V_i given in Theorem 3.1 (i).

M is isomorphic to either S^2 or $\mathbb{R}P^2$. If there exists an i $(1 \le i \le k)$ such that $\dim V_i$ *is an odd integer then M is isomorphic to S^2 and vice versa.*

Proof. Take a basis X_2, X_3 of $\mathfrak{m}$ satisfying (7). Since M is isomorphic to $\mathbb{R}P^2$ if and only if $\exp(\pi/2\, X_2) \in SO(n)$, the assertion is obvious. □

EXAMPLE 10. Let $P = SU(4)/SO(4)$. The subspaces $\mathfrak{m}_1 = \mathbb{R}X_2 + \mathbb{R}X_{3,1}$ and $\mathfrak{m}_2 = \mathbb{R}X_2 + \mathbb{R}X_{3,-1}$ are Lie triple systems of $\mathfrak{p} = (\mathfrak{so}(4))^\perp$, where X_2, $X_{3,1}$ and $X_{3,-1}$ are those defined in Example 1. As it was shown that $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are not congruent under the action of $SO(4)$. But, in each case, the sectional curvature of the corresponding totally geodesic surface is equal to $1/5$.

EXAMPLE 11. Let $P = SU(8)/SO(8)$. Put

$$\widetilde{X_2} = \left[\begin{array}{c|c} X_2 & 0 \\ \hline 0 & X_2 \end{array}\right], \quad \widetilde{X_3} = \left[\begin{array}{c|c} X_{3,1} & 0 \\ \hline 0 & X_{3,1} \end{array}\right], \quad \widetilde{Y_2} = \left[\begin{array}{c|c} Y_2 & 0 \\ \hline 0 & 0 \end{array}\right], \quad \widetilde{Y_3} = \left[\begin{array}{c|c} Y_3 & 0 \\ \hline 0 & 0 \end{array}\right]$$

where X_2 and $X_{3,1}$ are those defined in exmaple 1 and

$$Y_2 = \sqrt{-1}\begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix}, \quad Y_3 = -\sqrt{-1}\begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

Followings are Lie triple systems;

(i) $\mathfrak{m}_1 = \mathbb{R}\widetilde{X_2} + \widetilde{X_3}$,

(ii) $\mathfrak{m}_2 = \mathbb{R}\widetilde{Y_2} + \widetilde{Y_3}$.

In either case, the sectional curvature of the corresponding totally geodesic surface is equal to $1/10$. The totally geodesic surface corresponding to $\mathfrak{m}_1$ is isomorphic to S^2 and the totally geodesic surface corresponding to $\mathfrak{m}_2$ is isomorphic to $\mathbb{R}P^2$.

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Faculty of Science and Engineering
Hosei University
3–7–2 Kajino-cho, Koganei, Tokyo
Japan
e-mail: mashimo@hosei.ac.jp